Problemen

| Problem Section

Edition 2007/4

We received submissions from Daniël Worm (Leiden), Kee-Wai Lau (Hong Kong), the Fejéntaláltuka Szeged Problem Group (Szeged), Eric Kuisch (Houten), Marcelina Mocanu (Bacău), Cornel Berceanu (Bacău), Sander Kupers (Utrecht), Sep Thijssen (Nijmegen), Jaap Spies, Rob van der Waall (Huizen), and H. Reuvers (Maastricht).

Problem 2007/4-A Let p be a prime number. Determine all n such that in the binomial formula

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

none of the coefficients is divisible by p.

Solution We present a solution similar to the one submitted by Sander Kupers. This problem was also solved by Kee-Wai Lau, the Fejéntaláltuka Szeged Problem Group, Eric Kuisch, Marcelina Mocanu, Cornel Berceanu, Sep Thijssen, Rob van der Waall and H. Reuvers.
The condition is satisfied by $n < p$.
Assume that $n \geq p$ and write $n = a_0 + a_1 p + \cdots + a_k p^k$, with $0 \leq a_i < p$, $a_k > 0$ and $k > 0$. Note that since $(x+1)^p \equiv x^p + 1$ modulo p, we have

$$(x+1)^n \equiv (x+1)^{a_0}(x^p+1)^{a_1} \cdots (x^{p^k}+1)^{a_k}.$$

For the coefficient of x^{p^k-1} to be non-zero (modulo p) we need that $a_i = p-1$ for all $i < k$. In fact, this is sufficient for all the coefficients to be non-zero, as we now show. Consider the coefficient (modulo p) of x^m, with $1 < m < n$, in

$$(x+1)^n \equiv (x+1)^{p-1}(x^p+1)^{p-1} \cdots (x^{p^{k-1}}+1)^{p-1}(x^{p^k}+1)^{a_k}.$$

Writing $m = b_0 + b_1 p + \cdots + b_k p^k$ with $0 \leq b_i < p$, we find that the coefficient of x^m is the product of the coefficients of $x^{p^i b_i}$ in $(x^{p^i}+1)^{p-1}$, over all i. Since none of these are 0 modulo p, the coefficient of x^m is non-zero modulo p.
We conclude that n satisfies the required condition if and only if $n = ap^k - 1$ for some $k \geq 0$ and $0 < a < p$.

Problem 2007/4-B Determine all positive real numbers a for which there exists a function $f : \mathbf{R}_{>0} \to \mathbf{R}_{>0}$ such that the inequality

$$f(x+\delta) > \delta f(x)^a$$

holds for all x and for all $\delta > 0$.

Solution We present a solution based on that of Daniël Worm. This problem was also solved by Sander Kupers.
Note that for $a \leq 1$ the function $f(x) = e^x$ satisfies

$$f(x+\delta) > \delta f(x)^a. \tag{1}$$

So from now on we assume $a > 1$. Let $f : \mathbf{R}_{>0} \to \mathbf{R}_{>0}$ be a function that satisfies (1). Take $n \in \mathbf{Z}_{>0}$ and put

$$\delta_i = \frac{a^{i-1}}{a^n - 1} \quad \text{for } i = 1, \ldots, n.$$

Then we obtain the inequality

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$$f(x+\frac{1}{a-1}) = f(x+\delta_1+\cdots+\delta_n) > \delta_1(f(x+\delta_2+\cdots+\delta_n))^a > \cdots > Cf(x)^{a^n},$$

where

$$C = \prod_{i=1}^{n} \delta_i^{a^{i-1}} = \frac{\prod_{i=1}^{n} a^{(i-1)a^{i-1}}}{(a^n-1)^{(a^n-1)/(a-1)}} > a^p,$$

and

$$p = \sum_{i=1}^{n}(i-1)a^{i-1} - n(a^n-1)/(a-1).$$

This exponent satisfies

$$p = \sum_{i=1}^{n}(i-1-n)a^{i-1} = -a^n \sum_{k=1}^{n} k\left(\frac{1}{a}\right)^k > -a^n \sum_{k=1}^{\infty} k\left(\frac{1}{a}\right)^k = a^n D$$

with $D \in \mathbf{R}$ (independent of n) as this infinite series converges. In conclusion we see that for all n and x we have

$$f(x+\frac{1}{a-1}) > \left(a^D f(x)\right)^{a^n}. \tag{2}$$

From (1) it follows that there exists x with $a^D f(x) > 1$, leading to a contradiction with (2). So we see that there exists a function $f : \mathbf{R}_{>0} \to \mathbf{R}_{>0}$ satisfying (1) if and only if $a \leq 1$.

Problem 2007/4-C Let G be a finite group with n elements. Let c be the number of pairs $(g_1, g_2) \in G \times G$ such that $g_1 g_2 = g_2 g_1$. Show that either G is commutative or that $8c \leq 5n^2$. Show that if $8c = 5n^2$ then 8 divides n.

Solution We received correct solutions by Jaap Spies and Rob van der Waall. We present a solution based on both submissions.

For $x \in G$ denote by n_x the number of $g \in G$ such that $gx = xg$. Denote by k the number of conjugacy classes of G. Then

$$c = \sum_{x\in G} n_x = kn,$$

where the first equality follows from the definition of c and the second by counting the orbits of the action of G on itself by conjugation. Hence $8c = 5n^2$ implies $8k = 5n$ and therefore that 8 divides n.

Assume now that G is not commutative. Denote the center of G by Z.

Claim. The quotient group G/Z is not cyclic.

Proof. Assume that gZ generates the quotient, then every element of G can be written as $g^i z$ for some i and some $z \in Z$, and it follows that all elements commute with each other, a contradiction since we assumed G not to be commutative.

So the index of Z in G is at least 4. Denote this index by a. Now we estimate k, the number of conjugacy classes of G. Every element of Z consitutes a single conjugacy class and every other conjugacy class must have at least two elements, hence

$$k \leq \frac{1}{a}n + \frac{1}{2}\frac{a-1}{a}n = \left(\frac{1}{2}+\frac{1}{2a}\right)n \leq \frac{5}{8}n$$

which is what we wanted to show.