Problemen

| Problem Section

Edition 2007/2

For Edition 2007/2 we received submissions from Kee-Wai Lau, Ronald Kortram, Hans Montanus, H.F.H. Reuvers, Lieke de Rooij, and Arne Smeets.

Problem 2007/2-A

1. Find the largest number c such that all natural numbers n satisfy $n\sqrt{2} - \lfloor n\sqrt{2}\rfloor \geq \frac{c}{n}$.
2. For this c, find all natural numbers n such that $n\sqrt{2} - \lfloor n\sqrt{2}\rfloor = \frac{c}{n}$.

Solution This problem was solved by Kee-Wai Lau, Hans Montanus, and Arne Smeets. The solution below is based on that of Kee-Wai Lau.

We first show that $n\sqrt{2} - \lfloor n\sqrt{2}\rfloor > \sqrt{2}/(4n)$ for all natural numbers n. Since $\sqrt{2}$ is irrational, we have $n\sqrt{2} > \lfloor n\sqrt{2}\rfloor$, $2n^2 > \lfloor n\sqrt{2}\rfloor^2$ and so $2n^2 - \lfloor n\sqrt{2}\rfloor^2 \geq 1$. Hence

$$n\sqrt{2} - \lfloor n\sqrt{2}\rfloor = \frac{2n^2 - \lfloor n\sqrt{2}\rfloor^2}{n\sqrt{2} + \lfloor n\sqrt{2}\rfloor} \geq \frac{1}{n\sqrt{2} + \lfloor n\sqrt{2}\rfloor} > \frac{1}{n\sqrt{2} + n\sqrt{2}} = \frac{\sqrt{2}}{4n}.$$

Next we show that the constant $\sqrt{2}/4$ cannot be replaced by any larger number. For natural numbers m, let

$$a_m = \frac{(\sqrt{2}+1)^{2m-1} + (\sqrt{2}-1)^{2m-1}}{2\sqrt{2}}.$$

By using the binomial theorem we see that a_m and

$$a_m\sqrt{2} - (\sqrt{2}-1)^{2m-1} = \frac{(\sqrt{2}+1)^{2m-1} - (\sqrt{2}-1)^{2m-1}}{2}$$

are positive integers. Since $a_m\sqrt{2} - 1 < a_m\sqrt{2} - (\sqrt{2}-1)^{2m-1} < a_m\sqrt{2}$, we have $\lfloor a_m\sqrt{2}\rfloor = a_m\sqrt{2} - (\sqrt{2}-1)^{2m-1}$. Hence

$$2a_m^2 - \lfloor a_m\sqrt{2}\rfloor^2 = 2a_m^2 - \left(a_m\sqrt{2} - (\sqrt{2}-1)^{2m-1}\right)^2 = 1$$

and

$$a_m\left(a_m\sqrt{2} - \lfloor a_m\sqrt{2}\rfloor\right) = \frac{a_m}{a_m\sqrt{2} + \lfloor a_m\sqrt{2}\rfloor} = \frac{a_m}{2\sqrt{2}a_m - (\sqrt{2}-1)^{2m-1}},$$

which tends to $\sqrt{2}/4$ as m tends to infinity. This completes the solution.

Problem 2007/2-B Find polynomials $f(x)$ and $g(x)$ such that

$$\int_0^x \frac{6t\mathrm{d}t}{\sqrt{t^4+4t^3-6t^2+4t+1}} = \log\left(f(x) + g(x)\sqrt{x^4+4x^3-6x^2+4x+1}\right).$$

Solution This problem was solved by Ronald Kortram, Lieke de Rooij and Arne Smeets. The solution below is based on that of Arne Smeets.

Let

$$p(x) = x^4 + 4x^3 - 6x^2 + 4x + 1.$$

Deriving the given equality with respect to x and rewriting the result gives

$$\frac{6x}{\sqrt{p(x)}} = \frac{2\sqrt{p(x)}f'(x) + 2p(x)g'(x) + p'(x)g(x)}{2\sqrt{p(x)}f(x) + 2p(x)g(x)},$$

or, equivalently,

$$\left(f'(x) - 6xg(x)\right)2\sqrt{p(x)} + \left(2p(x)g'(x) + p'(x)g(x) - 12xf(x)\right) = 0.$$

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From this we can easily deduce, for example using the fact that $\mathbf{R}[x]$ is a unique factorization domain, that

$$f'(x) = 6xg(x),$$
$$2p(x)g'(x) = 12xf(x) - p'(x)g(x).$$

The first equation implies that $\deg f(x) - \deg g(x) = 2$. Let $n = \deg g(x)$ and let α and β be the coefficients of x^{n+2} and x^n in $f(x)$ and $g(x)$, respectively. The first equation implies that $(n+2)\alpha = 6\beta$, while the second implies that $2n\beta = 12\alpha - 4\beta$, or $(n+2)\beta = 6\alpha$. Consequently $n = 4$ and $\alpha = \beta$. Let

$$f(x) = a_6x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0,$$
$$g(x) = b_4x^4 + b_3x^3 + b_2x^2 + b_1x + b_0.$$

The two equations given above imply the following relations between the coefficients:

$$\begin{aligned}
6a_6 &= 6b_4\\
5a_5 &= 6b_3\\
4a_4 &= 6b_2\\
3a_3 &= 6b_1\\
2a_2 &= 6b_0\\
a_1 &= 0\\
32b_4 + 6b_3 &= 12a_5 - 12b_4 - 4b_3\\
-48b_4 + 24b_3 + 4b_2 &= 12a_4 + 12b_4 - 12b_3 - 4b_2\\
32b_4 - 36b_3 + 16b_2 + 2b_1 &= 12a_3 - 4b_4 + 12b_3 - 12b_2 - 4b_1\\
8b_4 + 24b_3 - 24b_2 + 8b_1 &= 12a_2 - 4b_3 + 12b_2 - 12b_1 - 4b_0\\
6b_3 + 16b_2 - 12b_1 &= 12a_1 - 4b_2 + 12b_1 - 12b_0\\
4b_2 + 8b_1 &= 12a_0 - 4b_1 + 12b_0\\
2b_1 &= -4b_0.
\end{aligned}$$

The solutions

$$(a_6, a_5, a_4, a_3, a_2, a_1, a_0, b_4, b_3, b_2, b_1, b_0)$$

of this system of linear equations are proportional to

$$(1, 12, 45, 44, -33, 0, 43, 1, 10, 30, 22, -11).$$

Finally, setting $x = 0$ leads to the condition $f(0) + g(0) = a_0 + b_0 = 1$, so that

$$f(x) = \frac{1}{32}\left(x^6 + 12x^5 + 45x^4 + 44x^3 - 33x^2 + 43\right),$$
$$g(x) = \frac{1}{32}\left(x^4 + 10x^3 + 30x^2 + 22x - 11\right).$$

On Problem B of NAW 5/8 nr. 2 juni 2007, by Bas Edixhoven

The aim of this short note is to give some indication of the interesting and well-known theoretical background of this problem. Let $h(x)$ denote the polynomial $x^4 + 4x^3 - 6x^2 + 4x + 1$. The problem is then to find polynomials $f(x)$ and $g(x)$ such that:

$$\int 6xh(x)^{-1/2}dx = \log(f(x) + g(x)h(x)^{1/2}).$$

The left-hand side is an example of an *elliptic integral*: the function to be integrated is a rational function in x and the square root of a polynomial of degree 3 or 4 in x. Only very special elliptic integrals can be expressed in terms of elementary functions as is the case here (the interested reader is advised to consult the Wikipedia page on this subject).

So, what makes this possible in the present case? The answer will be given in terms of algebraic geometry, and, in particular, of a Riemann surface (the possibly intimidated reader is kindly asked *not* to stop reading at this point).

We let E_0 be the solution set in $\mathbf{C}^2$ of the equation $y^2 = h(x)$. The projection from E_0 to $\mathbf{C}$ that sends (a, b) to a is a two-to-one map, except at the a's with $h(a) = 0$. Using Euclid's algorithm one finds that $h(x)$ and $h'(x)$ have no common zeros. This implies that E_0 is non singular: the gradient $(-h'(x), 2y)$ of the function $y^2 - h(x)$ has no common zero with $y^2 - h(x)$ itself. A complex analytic version of the implicit function theorem then shows that every point P of E_0 has an open neighborhood U that is analytically isomorphic to a small disk D around 0 in $\mathbf{C}$; any function $z\colon U \to D$ that gives such an isomorphism is then called a *coordinate* at P.

In the theory of analytic functions, one often completes (or *compactifies*) $\mathbf{C}$ (with coordinate x, say) to the so-called Riemann sphere $\mathbf{P}^1(\mathbf{C})$ by adding one point, called ∞. In fact, one takes another copy of $\mathbf{C}$ with coordinate u, say, and glues the two copies along their subsets $\mathbf{C} - \{0\}$ by identifying a in the x-copy with a^{-1} in the u-copy.

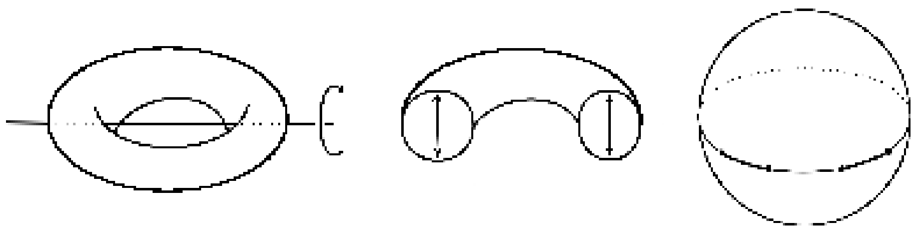
We want to complete E_0 in a similar way. In $(\mathbf{C} - \{0\}) \times \mathbf{C}$, E_0 is given by the equation

$$y^2 = h(x) = h(u^{-1}),$$

and hence also by $(u^2 y)^2 = u^4 h(u^{-1})$.

We let $k(u)$ be the polynomial $u^4 h(u^{-1})$ (this happens to be $h(u)$ but that is just a coincidence). We let E_∞ be the solution set in $\mathbf{C}^2$ (with coordinates u and v) of the equation $v^2 = k(u)$. The projection from E_∞ to $\mathbf{C}$ that sends (c, d) to c is two-to-one except at the four zeros of $k(u)$, and E_∞ is non-singular as well. We glue E_0 and E_∞ along their subsets where the first coordinate is non-zero by identifying (a, b) in E_0 with (a^{-1}, ba^{-2}) in E_∞. The result is a compact Riemann surface that we call E, with a map to $\mathbf{P}^1(\mathbf{C})$.

As an oriented surface, it can be seen that E is a torus. Such Riemann surfaces are called elliptic curves (the interested reader can again consult Wikipedia). Just as on the Riemann sphere, meromorphic functions are rational functions (quotients of functions given by polynomials in x and y (or in u and v)).



The three pictures show, topologically, how the Riemann surface E, a torus, is mapped to the Riemann sphere. The map is the quotient for the rotation over 180 degrees about the axis that is shown in the first picture. As this rotation interchanges the 'front half' and the 'back half' of E, the quotient is obtained by identifying the boundary points in the way shown in the second picture. This has the effect of closing the two ends of the cylinder, giving a (deformed) sphere. The third picture shows this sphere, with the images of the two boundary circles drawn a bit fatter, on the equator. The four endpoints of these two segments, drawn still fatter, are the four ramification points.

One also has the notion of meromorphic differential form. Such a form ω gives for each local coordinate $z\colon U \to D \subset \mathbf{C}$ a differential form $F dz$, with F a meromorphic function on U. For example, any meromorphic function F on E gives the form dF which in a local coordinate is just $F' dz$, where F' is the derivative of F with respect to z. In terms of power series, or, in fact, Laurent series, if $F = \sum_n F_n z^n$ then $F' = \sum_n nF_n z^{n-1}$; here the F_n are in $\mathbf{C}$, zero for n sufficiently negative.

Problem B can now be stated as follows: find a meromorphic function F on E such that $(dF)/F = (6x dx)/y$.

Let P be in E, and z a local coordinate at P. Then we can write, uniquely, in a neighborhood of P:

$$(6x dx)/y = z^{n_P} G dz = z^{n_P+1} G(dz)/z,$$

with $G = \sum_{n \geq 0} G_n z^n$ and $G_0 \neq 0$. The integer n_P is called the *order* of $(6x dx)/y$ at P, and the complex number G_{-1-n_P} is called the *residue* of $(6x dx)/y$ at P (if $-1 - n_P$ is negative, the residue is zero).

Suppose that $(6xdx)/y = (dF)/F$, with F a meromorphic function on E. For P in E, and z a local coordinate at P, we can write, uniquely, in a neighborhood of P:

$$F = z^{m_P}H, \quad \text{with } H = \sum_{n\geq 0} H_n z^n \text{ and } H_0 \neq 0,$$

and hence

$$(dF)/F = (dz^{m_P})/(z^{m_P}) + (dH)/H = m_P(dz)/z + (dH)/H.$$

As $H_0 \neq 0$, the order of $(dH)/H$ at P is ≥ 0, and so it follows that the order of $(dF)/F$ at P is -1 precisely at the P with $m_P \neq 0$, with residue m_P.
It is a standard exercise to find all P where n_P is negative. One finds that this happens at the points P_+ and P_- in E_∞ with $u(P_\pm) = 0$ and $v(P_\pm) = \pm 1$, and that $n_{P_\pm} = -1$. Indeed, as a coordinate at these points one can take the function u, and one simply computes:

$$(6xdx)/y = 6u^{-1}d(u^{-1})/(vu^{-2}) = (-6/v)(du)/u.$$

This means that F must have order 6 at P_-, order -6 at P_+, and no poles or zeros outside $\{P_+, P_-\}$. The theory of elliptic curves (see Wikipedia for more information) shows that such a function F, if it exists, is unique up to a multiplicative constant, and that the existence is equivalent to the difference $P_+ - P_-$, in the so-called *group law* of E, being of finite order, dividing 6. As this group is isomorphic to a product of two circles, we conclude that this property of $P_+ - P_-$ is very special indeed.
Let us end by mentioning that the explanation above does not make the calculations found in the solution of Problem B easier, but that it does help in understanding what is happening. In particular, such a calculation becomes more than just a manipulation of formulas; one can understand what one is doing. This year, the Dutch national mastermath program (see www.mastermath.nl) contains two courses on elliptic curves, one in the Fall of 2007 (late news, unfortunately) and one in the Spring of 2008.

Problem 2007/2-C Consider the following game with persons A and B. Player A receives a random number uniformly distributed between 0 and 1. Player B receives two random numbers uniformly distributed between 0 and 1, and chooses the highest one. Each player can then choose to discard his number and receive a new random number between 0 and 1, in order to get a higher number. This choice is made without knowing the other player's number or whether the other player chose to replace his number. The player with the highest number wins. What strategy should the players follow to ensure they will win the game? What is the probability that person B wins the game? See also domino.research.ibm.com/Comm/wwwr_ponder.nsf/challenges/Februari2007.html

Solution This problem was solved by Hans Montanus, H.F.H. Reuvers and Lieke de Rooij. The solution below is based on that of Hans Montanus.
Let g be the boundary under which player A chooses to discard his number. We have the following stochastic variables for player A: X is the first number, Y is the new number, if it exists, and V is the final choice. That is, $V = X$ if $X \geq g$ and $V = Y$ if $X < g$.
For $0 \leq v < g$, the distribution function of V is

$$\begin{aligned} f(v) &= \lim_{\Delta v \to 0} \frac{P(v < V < v + \Delta v)}{\Delta v} = \lim_{\Delta v \to 0} \frac{P(X < g) \cdot P(v < Y < v + \Delta v)}{\Delta v} \\ &= \lim_{\Delta v \to 0} \frac{g \cdot \Delta v}{\Delta v} = g. \end{aligned}$$

For $g < v \leq 1$, it is

$$\begin{aligned} f(v) &= \lim_{\Delta v \to 0} \frac{P(v < V < v + \Delta v)}{\Delta v} \\ &= \lim_{\Delta v \to 0} \frac{P(v < X < v + \Delta v) + P(X < g) \cdot P(v < Y < v + \Delta v)}{\Delta v} = 1 + g. \end{aligned}$$

Let h be the boundary under which player B chooses to discard his number. For player B we use the following stochastic variables: X is the first number, Y is the second number, Z is the new number, if it exists, and W is the final choice.

For $0 \le w < h$, the distribution function of W is

$$f(w) = \lim_{\Delta w \to 0} \frac{P(w < W < w + \Delta w)}{\Delta w} = \lim_{\Delta w \to 0} \frac{P(X < h) \cdot P(Y < h) \cdot P(w < Z < w + \Delta w)}{\Delta w} = h^2.$$

For $h < w \le 1$, it is

$$f(w) = \lim_{\Delta w \to 0} \{ \frac{P(X < w) \cdot P(w < Y < w + \Delta w)}{\Delta w} + \frac{P(Y < w) \cdot P(w < X < w + \Delta w)}{\Delta w} + \frac{P(X < h) \cdot P(Y < h) \cdot P(w < Z < w + \Delta w)}{\Delta w} \} = 2w + h^2.$$

If $h \ge g$, the probability $P_1(B)$ that player B wins is

$$P_1(B) = \int_0^1 P(V < w) f(w) dw = \int_0^g P(V < w) h^2 dw + \int_g^h P(V < w) h^2 dw + \int_h^1 P(V < w)(h^2 + 2w) dw.$$

Substituting

$$P(V < w | w < g) = \int_0^w f(v) dv = \int_0^w g dv = gw$$
$$\text{and } P(V < w | w > g) = \int_0^g g dv + \int_g^w (1 + g) dv = w - g + gw,$$

gives

$$P_1(B) = \int_0^g gwh^2 dw + \int_g^h (w - g + gw) h^2 dw + \int_h^1 (w - g + gw)(h^2 + 2w) dw = \frac{2}{3} - \frac{1}{3}g + \frac{1}{2}h^2 - \frac{2}{3}h^3 + \frac{1}{2}gh^2 - \frac{2}{3}gh^3 + \frac{1}{2}g^2h^2.$$

If $h \le g$, the probability $P_2(B)$ that player B wins is

$$P_2(B) = \int_0^1 P(V < w) f(w) dw = \int_0^h P(V < w) h^2 dw + \int_h^g P(V < w)(h^2 + 2w) dw + \int_g^1 P(V < w)(h^2 + 2w) dw.$$

Once more substituting

$$P(V < w | w < g) = gw \text{ and } P(V < w | w > g) = w - g + gw,$$

we find

$$P_2(B) = \int_0^h gwh^2 dw + \int_h^g gw(h^2 + 2w) dw + \int_g^1 (w - g + gw)(h^2 + 2w) dw = \frac{2}{3} - \frac{1}{3}g + \frac{1}{2}h^2 + \frac{1}{3}g^3 - \frac{1}{2}gh^2 - \frac{2}{3}gh^3 + \frac{1}{2}g^2h^2.$$

The probability $P(B)$ that player B wins satisfies $P(B) = P_1(B)$ if $h \ge g$ and $P(B) = P_2(B)$ if $h \le g$. For any value of g ($0 \le g \le 1$), $P(B)$ attains a maximum. If $h \ge g$, the maximum lies on the curve $h = (1 + g + g^2)/(2 + 2g)$ (using $\partial P_1(B)/\partial h = 0$); if $h \le g$, it lies on the curve $h = (1 - g + g^2)/(2g)$ (using $\partial P_2(B)/\partial h = 0$). These two curves meet at the point $(g, h) = ((-1 + \sqrt{5})/2, (-1 + \sqrt{5})/2)$. At this intersection point, $P_1(B)$ has a local minimum $(4 - \sqrt{5})/3$ on the curve $h = (1 + g + g^2)/(2 + 2g)$ $(0 < g \le (-1 + \sqrt{5})/2)$; $P_2(B)$ has an absolute minimum on the curve $h = (1 - g + g^2)/(2g)$ $((-1 + \sqrt{5})/2) \le g \le 1$, at a saddle point. This saddle point satisfies $\partial P_2(B)/\partial g = 0$, that is, $g^2 + gh^2 - \frac{1}{3} - \frac{1}{2}h^2 - \frac{2}{3}h^3 = 0$. Substituting $h = (1 - g + g^2)/(2g)$ leads to the equation

$$4g^6 + 15g^5 + 12g^4 - 15g^3 + 3g - 2 = 0.$$

The polynomial $4g^6 + 15g^5 + 12g^4 - 15g^3 + 3g - 2$ appears to be irreducible. A numerical approximation of the roots gives one positive real root, namely $g = 0,6488849...$, with corresponding value of h equal to $0,5949951...$.

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The strategy that the players should follow is this: player A asks for a new number if his first is less than $0,6488849...$ and player B asks for a new number if his first two are both less than $0,5949951...$. Substituting these values in the expression for $P_2(B)$ we find that the probability that player B wins is $0,587003...$.