Problemen

| Problem Section

Edition 2007/1

For Edition 2007/1 we received submissions from M.G. de Bruin, Martinus van Hoorn, Ruud Jeurissen, Marnix Klooster, A.J. Th. Maassen, Marco Pouw, H.F.H. Reuvers, Anton R. Schep, Jaap Spies, Fejéntaláltuka Szeged, Sep Thijssen, R.W. van den Waall, and Jianfu Wang.

Problem 2007/1-A Define the sequence $\{u_n\}$ by $u_1 = 1$, $u_{n+1} = 1 + (n/u_n)$. Prove or disprove that

$$u_n - 1 < \sqrt{n} \leq u_n.$$

Solution This problem was solved by Ruud Jeurissen, Marnix Klooster, A.J. Th. Maassen, Marco Pouw, H.F.H. Reuvers, Anton R. Schep, Jaap Spies, Fejéntaláltuka Szeged, Sep Thijssen, R.W. van den Waall, and Jianfu Wang. The solution below is based on that of Marco Pouw.

For $n = 1$ the statement is that $0 < 1 \leq 1$, which is correct. Suppose that for some $n \geq 1$ we have the following inequalities:

1. $u_{n+1} = 1 + n/u_n$
2. $u_n - 1 < \sqrt{n}$
3. $\sqrt{n} \leq u_n$

Then

$$u_{n+1} - 1 = \frac{n}{u_n} \leq \frac{n}{\sqrt{n}} = \sqrt{n} < \sqrt{n+1},$$

and

$$\begin{aligned} u_{n+1} - 1 = \frac{n}{u_n} &> \frac{n}{\sqrt{n+1}} \\ &> \frac{n}{\sqrt{n+1}+1} = \frac{n(\sqrt{n+1}-1)}{(\sqrt{n+1}-1)(\sqrt{n+1}+1)} \\ &= \frac{n(\sqrt{n+1}-1)}{\sqrt{n+1}^2 - 1^2} = \sqrt{n+1} - 1. \end{aligned}$$

Combining the two inequalities gives $u_{n+1} - 1 < \sqrt{n+1} \leq u_{n+1}$. We conclude by induction on n that the inequalities 1 through 3 hold for all $n \geq 1$.

Problem 2007/1-B Given a non-degenerate tetrahedron (whose vertices do not all lie in the same plane), which conditions have to be satisfied in order that the altitudes intersect at one point?

Solution This problem was solved by Martinus van Hoorn, Ruud Jeurissen, A.J.Th. Maassen, H.F.H. Reuvers, Jaap Spies, and R.W. van den Waall. The solution below is based on that of Ruud Jeurissen.

In a tetrahedron ABCD, the altitudes from A and B are both orthogonal to CD. If they intersect, say at S, then CD is orthogonal to the plane ABS, hence to AB.

Conversely, suppose that CD is orthogonal to AB. Take T on CD such that AT is the altitude from A in ΔACD. Then CD is orthogonal to the plane through A, B and T, hence also to BT. The altitudes from A and B, respectively, in ΔABT are then orthogonal to CD and to BT and AT, respectively, so must be the altitudes from A and B, respectively, in ABCD. The altitudes from A and B intersect in ΔABT.

Likewise, for the altitudes from A and C to intersect it is necessary and sufficient that AC and BD be perpendicular.

Eindredactie:
Lenny Taelman, Johan Bosman, Matthijs Coster, Reinie Erné
Redactieadres:
Problemenrubriek NAW
Mathematisch Instituut
Postbus 9512
2300 RA Leiden
uwc@nieuwarchief.nl

Next suppose that AB and CD are perpendicular to each other, and AC and BD as well. Then AD and BC are also perpendicular to each other (if for vectors a, b and c in a Euclidean space, the inner products $\langle b-a, c\rangle$ and $\langle c-a, b\rangle$ are 0, then $\langle b-c, a\rangle = 0$).
The altitude from C intersects both that from A and that from B. Both points of intersection are on the plane through A,B and T (as above), so they coincide with S (as above). The same holds for D instead of C.
We conclude that the altitudes of a tetrahedron intersect at one point if and only if there are two pairs of perpendicular altitudes.

Problem 2007/1-C Let e be a positive integer, and let d be an element of $\{0, 1, 2, \ldots, 3e\}$. Show that the polynomial

$$P = \sum_{\substack{a\geq 0, b\geq 0, c\geq 0\\ a+b+c=d}} \frac{d!}{a!b!c!}\binom{e}{a}\binom{e}{b}\binom{e}{c} x^a y^b z^c$$

in the three variables x, y, and z is not divisible by $x+y+z$ unless $d=1$.

Solution We did not receive any solutions to this problem. The solution below is the proposer's.
The proof is split into two parts. The case where $d \geq 2e$ is easy: P is symmetric in x, y, z, and hence equal to $Q(\sigma_1, \sigma_2, \sigma_3)$ for a unique integral polynomial Q, where $\sigma_1 := x+y+z$, $\sigma_2 := xy+xz+yz$, and $\sigma_3 := xyz$. The fact that the lexicographically largest monomial in P with respect to $x > y > z$ is $x^e y^e z^{d-e}$ implies that Q contains the monomial $\sigma_3^{d-2e}\sigma_2^{3e-d}$, which is not divisible by σ_1.
For $d < 2e$ we use that P is the coefficient of $X^e Y^e Z^e$ in

$$\frac{1}{d!}(X+Y+Z)^d \left(x\frac{\partial}{\partial X} + y\frac{\partial}{\partial Y} + z\frac{\partial}{\partial Z}\right)^d X^e Y^e Z^e.$$

Modulo $(x+y+z)$ the derivation $(x\partial/\partial X + y\partial/\partial Y + z\partial/\partial Z)^d$ and the operation of multiplying with $X+Y+Z$ commute, so that the above expression is congruent to $\frac{1}{d!}(x\partial/\partial X + y\partial/\partial Y + z\partial/\partial Z)^d (X+Y+Z)^d X^e Y^e Z^e$ modulo $x+y+z$. The coefficient of $X^e Y^e Z^e$ in this expression equals

$$(-1)^d \sum_{a+b+c=d} \frac{d!}{a!b!c!}\binom{-e-1}{a}\binom{-e-1}{b}\binom{-e-1}{c}.$$

Writing $P = P_e$, we thus find the functional equation

$$(-1)^d P_{-e-1} \equiv P_e \mod x+y+z.$$

Let $Q = Q(e)$ be the coefficient of $x^{d-1}y$ in $P_e(x, y, -x-y)$; this is a polynomial in $\mathbf{Z}[e]$ of degree $\leq d$. We are done if we can show that $Q(e)$ is non zero for all e with $2e > d$; we do this by constructing many other zeroes of Q. First, Q is not the zero polynomial; this small exercise uses $d \neq 1$ and we skip it here. Next take $e' \in \mathbf{Z}_{\geq 0}$ with $2e' < d-1$; we claim that $Q(e') = 0$. Indeed, the term

$$\frac{d!}{a!b!c!}\binom{e}{a}\binom{e}{b}\binom{e}{c} x^a y^b (-x-y)^c$$

in $P_e(x, y, -x-y)$ can only contribute to Q if $a+c \geq d-1$. But then we find that at least one of a and c is $> e'$ so that one of the binomial coefficient vanishes. Finally, Q inherits the functional equation from P:

$$Q(-e-1) = (-1)^d Q(e),$$

which means that the zeroes of Q lie symmetrically around $-1/2$. This shows that if $d = 2l$ is even, then the zeroes of Q are $-l, -l+1, \ldots, l-1$, and if $d = 2l+1$, then Q has (at most) one more zero, at $-1/2$. We conclude that $Q(e) \neq 0$ for $e > d/2$ and we are done.