Problemen

| Problem Section

Edition 2006/4

For Edition 2006/4 We received submissions from J. Buskes, Birgit van Dalen, Maarten Derickx, Ruud Jeurissen, Ronald Kortram, Floor van Lamoen, Louis Maassen, H.F.H. Reuvers, Jaap Spies, Arjen Stolk, Rohith Varma, Hendrik Verhoek, Koen Vervloesem, Rob van der Waall.

Problem 2006/4-A Seventeen students play in a tournament featuring three sports: badminton, squash, and tennis. Any two students play against each other in exactly one of the three sports. Show that there is a group of at least three students who compete amongst themselves in one and the same sport.

Solution This problem was solved by Birgit van Dalen, Maarten Derickx, Ruud Jeurissen, Ronald Kortram, Louis Maassen, H.F.H. Reuvers, Arjen Stolk, Hendrik Verhoek, Koen Vervloesem. The solution below is based on that of Birgit van Dalen.

Lemma *Suppose that for any group of n students playing k different sports there is always a subgroup of three students who compete amongst themselves in the same sport. We can show that in any group of $(n-1)(k+1)+2$ students playing $k+1$ different sports there always exist a subgroup of three students who compete amongst themselves in the same sport.*

Proof To show this consider one of the $(n-1)(k+1)+2$ students. He plays against each of the $(n-1)(k+1)+1$ other students in one of the $k+1$ sports. Using the pigeonhole principle, there must be a sport he plays against at least n students. If two students in this subgroup of n students compete against each other in this same sport, we have found the triple we were looking for. Otherwise these n students play k different sports, and by assumption there is a subgroup of three students who compete amongst themselves in the same sport.

Any group of three students who compete in only one sport contains a subgroup of three students who compete amongst themselves in the same sport. Using the lemma, we derive that any group of $2\cdot 2+2=6$ students who play two sports contains a subgroup of three students who compete amongst themselves in one sport. Applying the lemma once more, we find that a group of $5\cdot 3+2=17$ students playing three sports contains a subgroup of three students who compete amongst themselves in the same sport.
This problem establishes that the Ramsey number $R_2(3,3,3)$ is at most 17.

Problem 2006/4-B The sequence $\{a_n\}_{n\geq 1}$ is defined by

$$a_1 = 1; a_2 = 12; a_3 = 20; a_{n+3} = 2a_{n+2} + 2a_{n+1} - a_n \; (n \in \mathbf{N}).$$

Prove that $4a_na_{n+1}+1$ is a square for all $n \in \mathbf{N}$.

Solution This problem was solved by J. Buskes, Birgit van Dalen, Maarten Derickx, Ronald Kortram, Floor van Lamoen, Louis Maassen, H.F.H. Reuvers, Jaap Spies, Arjen Stolk, Rohith Varma, Hendrik Verhoek. The solution below is based on that of Floor van Lamoen.

Let $b_n = a_n^2 + a_{n-1}^2 + a_{n-2}^2 - 2(a_na_{n-1} + a_na_{n-2} + a_{n-1}a_{n-2})$. By induction we will show that $b_n = 1$, or equivalently

$$\begin{aligned} 4a_na_{n-1}+1 &= 4a_na_{n-1}+b_n = a_n^2+a_{n-1}^2+a_{n-2}^2+2(a_na_{n-1}-a_na_{n-2}-a_{n-1}a_{n-2}) \\ &= (a_n+a_{n-1}-a_{n-2})^2, \end{aligned} \tag{1}$$

which is clearly true for $n=2$. Suppose $b_m=1$, so (1) holds for $n=m$ and $4a_ma_{m-1}+1 = (a_m+a_{m-1}-a_{m-2})^2$. By the recurrence relation for $\{a_n\}$ we see that $a_m+a_{m-1}-a_{m-2} = a_{m+1}-a_m-a_{m-1}$ so we have

$$4a_ma_{m-1}+1 = (a_{m+1}-a_m-a_{m-1})^2,$$

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which is equivalent to $b_{m+1} = 1$.
Note that the recurrence for going downwards through the sequence is

$$a_{n-3} = 2a_{n-2} + 2a_{n-1} - a_n,$$

so that $b_n = 1$ and (1) hold for the sequence extended in both directions, i.e., for all integers n.

Problem 2006/4-C Let G be a finite group of order $p+1$ with p an odd prime. Show that p divides the order of $Aut(G)$ if and only if p is a Mersenne prime, that is, of the form $2^n - 1$, and G is isomorphic to $(\mathbf{Z}/2)^n$.

Solution This problem was solved by Birgit van Dalen, Ruud Jeurissen, Ronald Kortram, Jaap Spies, Arjen Stolk, Rohith Varma, Hendrik Verhoek, Rob van der Waall. The solution below is based on that of Birgit van Dalen.

We suppose that p is odd, and therefore $|G|$ is even.

(1) If p divides the order of Aut (G), then p is a Mersenne prime and G is isomorphic to $(\mathbf{Z}/2\mathbf{Z})^n$.

Suppose that p divides the order of Aut (G). Then by Sylow's Theorem Aut (G) contains an element of order p, since p is prime. We may consider Aut (G) as a subgroup of the permutation group S_{p+1}. The element of order p can then be considered as a cycle of length p. The identity element of G is mapped to itself by every element of Aut (G). Consequently the cycle of length p contains all other elements of G. Therefore every element of G other than the identity can be mapped onto any of the other non-identity elements by an automorphism of G. This implies that all these elements of G must have the same order.

Since p is odd, the order of G is even. Hence there is an element in G of order 2. But then all elements of G have to be of order 2. The group G has to be Abelian, because $(xy)^2 = 1$, that is, $xyxy = 1$, hence $xy = yx$. Every finite Abelian group is isomorphic to a direct product of cyclic groups. In our case the order of any element is 2. Therefore the cyclic groups are $\mathbf{Z}/2\mathbf{Z}$ and $G = (\mathbf{Z}/2\mathbf{Z})^n$ for some n. We find that the order of G is 2^n, hence $p = 2^n - 1$, which is a Mersenne prime.

(2) If p is a Mersenne prime and $G = (\mathbf{Z}/2\mathbf{Z})^n$ then p divides the order of Aut (G).

We consider G as a vector space over $\mathbf{F}_2$. Choose a basis for G. An automorphism of G maps the basis onto a set of n linearly independent elements of G. Each map onto a set of n linearly independent elements corresponds to one automorphism. We will construct all the automorphisms as follows: One by one, we map each element of the basis onto a vector that is linearly independent of the images of the prior elements. In particular, the first element is sent to a non-zero vector. We find that the order of Aut (G) is

$$(2^n - 1)(2^n - 2)(2^n - 4)\dots(2^n - 2^{n-1}).$$

Therefore the order of Aut (G) is divisible by $p = 2^n - 1$.

Birgit van Dalen, Rob van der Waall, and Arjen Stolk all remark that the problem was not entirely correct since G could be the cyclic group of three elements with Aut $(G) = \mathbf{Z}/2\mathbf{Z}$, whereas 2 is not a Mersenne prime.