Problemen | Problem Section

Edition 2006/2
For Session 2006/2 We received submissions from Toon Brans, Ruud Jeurissen, Jaap Spies, and Peter Vandendriessche.

Problem 2006/2-A Prove or disprove the following:
In a 9×9 Sudoku–square one randomly places the numbers $1 \dots 8$. There is at least one field such that if any one of the numbers $1 \dots 9$ is placed there, the Sudoku–square can be filled in to a (not necessarily unique) complete solution.

Solution This problem was solved by Toon Brans, Ruud Jeurissen, and Peter Vandendriessche. Their solutions were comparable.
We disprove the statement as follows: If the numbers $1 \dots 8$ are placed on the main diagonal from the top left corner towards the bottom right, there is no field where a number between 1 and 8 can be placed without giving a contradiction. In the first row the 1 cannot be placed, in the second row no 2 is possible, ..., in the eighth row no 8. This is the case for the columns as well. In the field in the lower right corner no 8 is possible.

Problem 2006/2-B Imagine a flea circus consisting of n boxes in a row, numbered 1, 2,..., n. In each of the first m boxes there is one flea ($m \leq n$). Each flea can jump forward to boxes at a distance of at most $d = n - m$. For all fleas all d+1 jumps have the same probability.
The director of the circus has marked m boxes as special targets. On his sign all m fleas jump simultaneously (no collisions).
1. Calculate the probability that after the jump exactly m boxes are occupied.
2. Calculate the probability that all m marked boxes are occupied.

Solution No solutions were sent in. The solution below is based on that of the proposer Jaap Spies.

Part 1
The jumps of the fleas correspond to a bipartite graph G. The possible jumps of the fleas can be coded in a (0,1)-matrix B of size m by n, with $b_{ij} = 1$ if and only if $i \leq j \leq i + d$. The total number of jumps with exactly m boxes occupied is the same as the number of injective maps $\pi : \{1, \dots, m\} \to \{1, \dots, n\}$ for which $b_{i\pi(i)} = 1$ for all i. This number equals $\operatorname{per}(B)$, the permanent of B, see [1], p. 44. The probability that after the jump exactly m boxes are occupied is therefore $\operatorname{per}(B)/(d+1)^m$.

Part 2
Let A be the set of marked boxes, then $A = \{a_1, a_2, \dots, a_m\}$ is a subset of $\{1, 2, \dots, n\}$ with $1 \leq a_1 < a_2 \cdots < a_m \leq n$ and $0 < m \leq n$. A successful jump of the fleas can be associated with a bijection $\pi : \{1, \dots, m\} \to A$ such that $i \leq \pi(i) \leq i + d$ for all i. The number of such successful jumps is equal to the permanent of the $(0, 1)$-matrix C, of size m by m, defined by $c_{ij} = 1$ if and only if $i \leq a_j \leq i + d$. The probability that after the jump the m marked boxes are occupied is $\operatorname{per}(C)/(d+1)^m$.

References
[1] Brualdi, H.J. Ryser, Combinatorial Matrix Theory, Cambridge University Press, 1991.
[2] The Dancing School Problems: http://www.jaapspies.nl/mathfiles/problems.html

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Problem 2006/2-C We are given two measurable spaces $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ plus a sub-σ-algebra $\mathcal{C}$ of $\mathcal{A}$. We are also given a real-valued function f on $X \times Y$ that is measurable with respect to the σ-algebra $\mathcal{A} \otimes \mathcal{B}$ (generated by the family $\{A \times B : A \in \mathcal{A}, B \in \mathcal{B}\}$). Furthermore, each horizontal section f_y is measurable on X with respect to $\mathcal{C}$. Prove or disprove: f is measurable with respect to $\mathcal{C} \otimes \mathcal{B}$.

Solution No solutions were sent in. The solution below is based on that of the proposer Klaas Pieter Hart.
Here is a counterexample. Let $X = Y = \omega_1$, the first uncountable ordinal. Let $\mathcal{A} = \mathcal{B} = \mathcal{P}(\omega_1)$, the power set. Let $\mathcal{C}$ be the σ-algebra that consists of the countable subsets of ω_1 and those with countable complements.
The set $L = \{(\alpha, \beta) : \alpha \le \beta < \omega_1\}$ belongs to $\mathcal{A} \otimes \mathcal{B}$ but not to $\mathcal{C} \otimes \mathcal{B}$. Its horizontal sections of L are countable — they are of the form $[0, \beta]$ — and therefore they belong to $\mathcal{C}$. This means that the characteristic function of L is a counterexample.

L belongs to $\mathcal{A} \otimes \mathcal{B}$
To show that L belongs to $\mathcal{A} \otimes \mathcal{B}$ we construct a sequence $\langle A_n \times B_n : n \in \mathbf{N} \rangle$ of rectangles such that $L = \bigcap_m \bigcup_{n \ge m} A_n \times B_n$. This is done quite indirectly.
By recursion we construct a sequence $\langle X_\alpha : \alpha < \omega_1 \rangle$ of infinite subsets of $\mathbf{N}$ with the following property: if $\beta < \alpha$ then $X_\alpha \setminus X_\beta$ is finite and $X_\beta \setminus X_\alpha$ is infinite — we abbreviate this as $X_\alpha \subset^* X_\beta$. For $\alpha < \omega_1$ we write $Y_\alpha = \mathbf{N} \setminus X_{\alpha+1}$ and we observe that the two sequences obtained in this way satisfy

$$\alpha \le \beta \text{ if and only if } X_\alpha \cap Y_\beta \text{ is infinite.} \qquad (*)$$

From the X_α and Y_α we define $A_n = \{\alpha : n \in X_\alpha\}$ and $B_n = \{\alpha : n \in Y_\alpha\}$. The equivalence $(*)$ is translated in terms of the A_n and B_n reads

$$(\alpha, \beta) \in L \text{ if and only if } (\alpha, \beta) \in A_n \times B_n \text{ infinitely often.}$$

But this means $L = \bigcap_m \bigcup_{n \geq m} A_n \times B_n$, as promised.
It remains to construct the X_α. Start with $X_0 = \mathbf{N}$. At successor stages let $X_{\alpha+1}$ be the even numbered elements of X_α (in its monotone enumeration). If α is a limit ordinal enumerate $\{\beta : \beta < \alpha\}$ in a simple sequence $\{\beta_n : n \in \mathbf{N}\}$ and recursively let x_n be the minimum of $\bigcap_{i \leq n} X_{\beta_n} \setminus \{x_i : i < n\}$; then let $X_\alpha = \{x_n : n \in \mathbf{N}\}$.

L does not belong to $\mathcal{C} \otimes \mathcal{B}$
Let $\mathcal{D}$ denote the family of those subsets Z of $\omega_1 \times \omega_1$ for which one can find an ordinal α and a subset A of ω_1 such that $Z \cap \big([\alpha, \omega_1) \times \omega_1\big) = [\alpha, \omega_1) \times A$. We show that $\mathcal{D}$ is a σ-algebra that contains all rectangles $C \times B$ with $C \in \mathcal{C}$ and $B \in \mathcal{B}$; this implies $\mathcal{C} \otimes \mathcal{B} \subseteq \mathcal{D}$. Clearly L does not belong to $\mathcal{D}$ and hence not to $\mathcal{C} \otimes \mathcal{B}$.

$\mathcal{D}$ is a σ-algebra Clearly $\omega_1 \times \omega_1$ belongs to $\mathcal{D}$.
If Z belongs to $\mathcal{D}$ then so does its complement Z^c: if $Z \cap \big([\alpha, \omega_1) \times \omega_1\big) = [\alpha, \omega_1) \times A$ then $Z^c \cap \big([\alpha, \omega_1) \times \omega_1\big) = [\alpha, \omega_1) \times A^c$.
Let $\langle Z_n \rangle_n$ be a sequence of elements of $\mathcal{D}$, with associated ordinals α_n and sets A_n. Let $\alpha = \sup_n \alpha_n$ and $A = \bigcup_n A_n$; then $\bigcup_n Z_n \cap \big([\alpha, \omega_1) \times \omega_1\big) = [\alpha, \omega_1) \times A$, so $\bigcup_n Z_n$ belongs to $\mathcal{D}$.

Every rectangle $C \times B$, with $C \in \mathcal{C}$ and $B \in \mathcal{B}$, belongs to $\mathcal{D}$ If C is countable let α be such that $C \subseteq [0, \alpha)$ then $C \times B \cap \big([\alpha, \omega_1) \times \omega_1\big) = [\alpha, \omega_1) \times \emptyset$.
If C is co-countable let α be such that $C^c \subseteq [0, \alpha)$ then $C \times B \cap \big([\alpha, \omega_1) \times \omega_1\big) = [\alpha, \omega_1) \times B$.

Remarks
In fact one has $\mathcal{D} = \mathcal{C} \otimes \mathcal{B}$. To see this let $Z \in \mathcal{D}$, with associated ordinal α and set A. Now observe that Z is the union of $[\alpha, \omega_1) \times A$ and $\bigcup_{\beta < \alpha} \{\beta\} \times Z_\beta$, where Z_β denotes $\{\gamma : (\beta, \gamma) \in Z\}$. This expresses Z as a countable union of rectangles from $\mathcal{C} \otimes \mathcal{B}$.
In Kunen's thesis [1] it is shown that, in fact, $\mathcal{A} \otimes \mathcal{B}$ is the whole power set of $\omega_1 \times \omega_1$; the proof that L belongs to the algebra is a simplification, for this special case, of Kunen's argument. The algebras $\mathcal{C}$ and $\mathcal{A}$ are quite far apart and rather extreme. I do not know what the answer is for more familiar σ-algebras. Specifically: can one prove that f is $\mathcal{C} \otimes \mathcal{B}$-measurable when both X and Y both are the real line, $\mathcal{A}$ and $\mathcal{B}$ the σ-algebra of Lebesgue-measurable sets and $\mathcal{C}$ the σ-algebra of Borel sets?

Reference
[1] Kunen, Kenneth, *Inaccessibility properties of cardinals*, Ph.D. thesis, Stanford University, 1968.