Problemen/UWC

Universitaire Wiskunde Competitie

Edition 2005/4
For Session 2005/4 we received submissions from Peter Vandendriessche, Vladislav Frank, Arne Smeets, Jan van de Lune, en P.G. Kluit.

Problem 2005/4-A We have $\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1$. Consequently partial sums must satisfy

$$\sum_{k \in K} \frac{1}{k(k+1)} < 1.$$

Show that for every $q \in \mathbf{Q}$ satisfying $0 < q < 1$, there exists a finite subset $K \subseteq \mathbf{N}$ so that

$$\sum_{k \in K} \frac{1}{k(k+1)} = q.$$

Solution This problem was solved by Peter Vandendriessche, Vladislav Frank and Arne Smeets. The solution below is based on that of Vladislav Frank.
First note that $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$. Hence $\frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} + \ldots + \frac{1}{(k+n-1)(k+n)} = \frac{1}{k} - \frac{1}{k+1} + \frac{1}{k+1} - \ldots + \frac{1}{k+n-1} - \frac{1}{k+n} = \frac{1}{k} - \frac{1}{k+n}$. Consequently it suffices to represent every rational number between 0 and 1 as $\frac{1}{a_1} - \frac{1}{a_2} + \ldots - \frac{1}{a_{2k}}$, where $a_1 \leq a_2 \leq a_3 \ldots \leq a_{2k}$. If two consecutive numbers are equal, they simply cancel out, so we allow equal numbers. This will be useful in final step of proof.
Let $\frac{a}{b}$ be our rational number. There is a natural number n such that $\frac{1}{n+1} < \frac{a}{b} \leq \frac{1}{n}$. Consider $x = \frac{1}{n} - \frac{a}{b} = \frac{b-an}{bn}$. The numerator of this fraction is non-negative because $\frac{a}{b} \leq \frac{1}{n}$, but less than a, the numerator of $\frac{a}{b}$, because $b - a(n+1) < 0$.
We have $\frac{a}{b} = \frac{1}{n} - x$. We now apply the same algorithm to x. Let m be a natural number such that $\frac{1}{m+1} < x \leq \frac{1}{m}$. The claim is that $m \geq n$. Namely, $x = \frac{b-an}{bn} < \frac{a}{bn} \leq \frac{1}{n^2}$, hence $m \geq n^2 + 1 > n$.
If we continue this algorithm, we obtain $\frac{a}{b} = \frac{1}{a_1} - (\frac{1}{a_2} - (\frac{1}{a_3} - (\ldots - (\frac{1}{a_x})\ldots)))$. Notice that the algorithm can only be repeated finitely many times, as the numerator decreases at each step. We now have $\frac{a}{b} = \frac{1}{a_1} - \frac{1}{a_2} + \frac{1}{a_3} - \ldots \pm \frac{1}{a_x}$. If x is even we are done.
In other case we may assume that $a_x > a_{x-1}$ and change $\frac{1}{a_x}$ into $\frac{1}{a_x - 1} - \frac{1}{a_x(a_x-1)}$. Here $a_1 \leq a_2 \leq \ldots \leq a_{x-1} \leq a_x - 1 \leq a_x(a_x - 1)$ and we are done. Of course $a_x \neq 1$, as otherwise $\frac{a}{b} = \frac{1}{1} = 1$ which is impossible.
As a generalization, V. Frank shows that for any irrational number in the interval [0,1] there exists an infinite sum.

Problem 2005/4-B We consider the progressive arithmetic and geometric means of the function sequence $f_n(x) = x^{n-1}, n \in \mathbf{N}, x > 0, x \neq 1$. These are

$$A_n = A_n(x) = \frac{1}{n}(1 + x + x^2 + \cdots + x^{n-1}) = \frac{x^n - 1}{n(x-1)}$$

and

$$G_n = G_n(x) = (x^{1+2+\cdots+(n-1)})^{\frac{1}{n}} = x^{\frac{n-1}{2}}.$$

The *Martins-property* reads $A_{n+1}/A_n \geq G_{n+1}/G_n$. In our case this gives

$$\frac{n}{n+1}\frac{x^{n+1}-1}{x^n - 1} \geq \sqrt{x}.$$

Prove, more generally, that

$$\frac{a}{a+1}\frac{x^{a+1}-1}{x^a - 1} \geq \sqrt{x} \text{ for } a > -\frac{1}{2}, x > 0, x \neq 1.$$

Solution This problem was solved by Jan van de Lune, Peter Vandendriessche, Vladislav Frank and Arne Smeets. The solution below is based on that of Peter Vandendriessche.

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Let $f(t)$ be a (smooth) non-negative function that is convex on $[a,b]$ and let $[x,y] \subset [a,b]$ such that $x+y=a+b$. We then have

$$\frac{\int_a^b f(t)\mathrm{d}t}{b-a} \geq \frac{\int_x^y f(t)\mathrm{d}t}{y-x}\,.$$

To prove this, consider, for given $f(t)$, x, and y, the function $g(t)$ defined by

$$g(t) = f(x) + \frac{(f(y)-f(x))(t-x)}{y-x}\,.$$

$g(t)$ is the line through the points $(x, f(x))$ and $(y, f(y))$. Notice that the convexity of f gives $\int_x^y g(t)\mathrm{d}t \geq \int_x^y f(t)\mathrm{d}t$. Let

$$h(t) = g(t) - \int_x^y g(t)\mathrm{d}t + \int_x^y f(t)\mathrm{d}t\,,$$

then

$$\int_x^y h(t)\mathrm{d}t = \int_x^y f(t)\mathrm{d}t\,.$$

By convexity we have $f(t) \geq g(t) \geq h(t)$ for $t \in [a,x] \cap [y,b]$. Since $h(t)$ is the equation of a line and $x+y=a+b$, we have

$$\frac{\int_a^b h(t)\mathrm{d}t}{b-a} = \frac{\int_x^y h(t)\mathrm{d}t}{y-x}\,.$$

Combining these results we find:

$$\begin{aligned}\frac{\int_a^b f(t)\mathrm{d}t}{b-a} &= \frac{\int_a^x f(t)\mathrm{d}t + \int_y^b f(t)\mathrm{d}t}{b-a} + \frac{\int_x^y f(t)\mathrm{d}t}{y-x}\\ &\geq \frac{\int_a^x h(t)\mathrm{d}t + \int_y^b h(t)\mathrm{d}t}{b-a} + \frac{\int_x^y h(t)\mathrm{d}t}{y-x}\\ &= \frac{\int_a^b h(t)\mathrm{d}t}{y-x} = \frac{\int_a^b f(t)\mathrm{d}t}{y-x}\,.\end{aligned}$$

Problem B is a special case of this result. For $x \in \mathbf{R}_0^+$, $x \neq 1$, let $f(t) = x^t$. Then $f''(t) = x^t \log^2(x) \geq 0$. Therefore $f(t)$ is convex. We have to distinguish two cases:

- $a \in (-\frac{1}{2}, 0)$. Then $0 < a + \frac{1}{2} < a+1$. Apply the lemma to the interval $[a+\frac{1}{2}, \frac{1}{2}] \subset [0, a+1]$.
- $a \in (0, \infty)$. Then $0 < \frac{1}{2} < a+\frac{1}{2} < a+1$. Apply the lemma to the interval $[\frac{1}{2}, a+\frac{1}{2}] \subset [0, a+1]$.

Notice that in the first case the sign in both numerator and denominator changes on the right side of the equation:

$$\frac{\int_0^{a+1} x^t \mathrm{d}t}{a+1} \geq \frac{\int_{\frac{1}{2}}^{a+\frac{1}{2}} x^t \mathrm{d}t}{a}\,,$$

from which we can deduce

$$\frac{a}{a+1} \cdot \frac{x^{a+1}-1}{\sqrt{x}\cdot(x^a-1)} \geq 1\,.$$

It is easy to prove the generalization

$$\frac{z-y}{a-b} \cdot \frac{x^{b-a}-1}{x^{z-y}-1} \geq x^{y-a}\,,$$

where $y+z=a+b$ and $-\frac{1}{2} < a < y < z < b$.

Problem 2005/4-C A *finite geometry* is a geometric system that has only a finite number of points. For an *affine plane geometry*, the axioms are as follows:

1. Given any two distinct points, there is exactly one line that includes both points.
2. The parallel postulate: Given a line L and a point P not on L, there exists exactly one line through P that is parallel to L.
3. There exists a set of four points, no three collinear.

We denote the set of points by $\mathbf{P}$, and the set of lines by $\mathbf{L}$. Let σ be an automorphism of $(\mathbf{P}, \mathbf{L})$ (meaning that three collinear points of $\mathbf{P}$ are mapped onto three collinear points of $\mathbf{P}$ and three non-collinear points of $\mathbf{P}$ are mapped onto three non-collinear points of $\mathbf{P}$). Prove that there exists a point $P \in \mathbf{P}$ with $\sigma(P) = P$ or a line $L \in \mathbf{L}$ with $\sigma(L) = L$ or $\sigma(L) \cap L = \emptyset$.

Solution This problem has been solved by Leendert Bleijenga and Peter Vandendriessche. The solution below is based on their solutions.
First we will prove the following lemma:

Lemma. *Let $M, L \in \mathbf{L}$, then $|M| = |L|$.*

Proof. Suppose that $|M \cap L| > 1$ then $M = L$. Therefore we may assume that $|M \cap L| = 1$. Let $|M| = m$ and $|L| = l$. By Axiom 3 we know that there exists a $P \in \mathbf{P}$ such that $P \notin L$ and $p \notin M$. Through P we can construct 1 line parallel to L and l lines that intersect L in its l points. In the same way we can construct, through P, 1 line parallel to M and m lines that intersect M in its m points. Let us now determine the number of lines through P; this equals $l + 1$ and $m + 1$. If $|M \cap L| = 0$, pick points $a \in L$ and $b \in M$. Let N be the line through a and b. Then by the previous argument $|L| = |N|$ and $|M| = |N|$. □

We conclude that all lines consist of an equal number of points, say s.

Lemma. $|\mathbf{P}| < |\mathbf{L}|$.

Proof. Let $|\mathbf{P}| = p$ and $|\mathbf{L}| = l$. Every two points define a line, and there are $\frac{1}{2}p(p-1)$ pairs of points. Each line has s points and is counted $\frac{1}{2}s(s-1)$ times. Therefore $l = \frac{p(p-1)}{s(s-1)}$. In order to show that $p < l$ we have to prove that $s(s-1) < p-1$ or $p > s^2 - s + 1$. The third axiom tells us that there exist three non-collinear points $a, b, c \in \mathbf{P}$. Let L be the line through a and b, M the line through a and c. By the parallel postulate, through every point on L there is exactly one line parallel to M. Starting with s points on L, we find s lines, all consisting of s points. Therefore $p \geq s^2$. □

Suppose that $\sigma(p) \neq p$, for all $p \in \mathbf{P}$, and that $\sigma(L) \neq L$ and $\sigma(L) \cap L \neq \emptyset$ for all $L \in \mathbf{L}$. Consider the function $\mu : \mathbf{L} \to \mathbf{P}$ given by $\mu(L) = \sigma(L) \cap L$. μ is well defined since $\sigma(L) \cap L$ is always a unique point. Now suppose that $\mu(L) = \mu(M)$ or $\sigma(L) \cap L = \sigma(M) \cap M = p$, and $\sigma(q) = p$. Then $q \in L$ and $q \in M$. We know that $q \neq p$. Therefore $L = M$ and μ is injective. However, if μ is injective, then $|\mathbf{P}| \geq |\mathbf{L}|$, which contradicts the previous lemma.

Problem 2005/4-* We have $\sum_{k=2}^{\infty} 1/k^2 = (\pi^2/6) - 1$. Consequently partial sums must satisfy

$$\sum_{k \in K} \frac{1}{k^2} < \frac{\pi^2}{6} - 1.$$

Given any $q \in \mathbf{Q}$ satisfying $0 < q < (\pi^2/6) - 1$, does there exist a finite subset $K \subseteq \mathbf{N} \setminus \{1\}$ so that

$$\sum_{k \in K} \frac{1}{k^2} = q?$$

Solution This problem was solved by P.G. Kluit. The solution below is based on his solution.
Let $q = \sum k_i^{-2}$, where k_i are different integers. Let m be the least common multiple of all k_i in the sum. For each such k_i a number k_i' exists such that $k_i k_i' = m$. Then $q = \frac{1}{m^2} \cdot \sum (k_i')^2$, that is, q can be written as a fraction with denominator m^2 and the numerator a sum of squares of different divisors of m. This raises the question: given m, which numbers can be written as sums of squares of different divisors of m? We will show that for highly composite numbers m, more specifically $m = n!$, the answer will be that sufficiently many integers can be written as sums of squares to prove the problem.

Lemma. *Let $n \geq 5$ be an integer and let $3 = d_1 < d_2 < \ldots d_m = n!/3$ be all divisors of $n!$ between 3 and $n!/3$. Then $2d_k^2 > d_{k+1}^2$ for $1 \leq k < m$.*

Proof. Let us prove this by induction. For $n = 5$ the divisors $d_1, \ldots, d_{12}$ are 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, and 40. It is easy to verify the lemma.
We assume the lemma is true for n. We have to prove that the Lemma holds for the divisors of $(n+1)!$. The divisors of $(n+1)!$ that are less than $n!/3$ clearly satisfy the lemma. Even though there may be more divisors, this cannot influence the inequality. Suppose that d_k and d_{k+1} are two successive divisors of $(n+1)!$, with $n!/3 \leq d_k < d_{k+1} \leq (n+1)!/3$. Let $d_k d'_k = d_{k+1} d'_{k+1} = (n+1)!$. Then d'_{k+1} and d'_k are two successive divisors of $(n+1)!$ with $3 \leq d'_{k+1} < d'_k \leq 3(n+1)$. As for $n \geq 5$ we have $3(n+1) < n!/3$, this suffices to conclude the proof. □

Lemma. *Let $n \in [129, 256]$ be an integer. Then n can be represented as a sum of different squares $d_1^2 + \ldots + d_k^2$, where $1 \leq d_1 < \ldots < d_k \leq 10$.*

Proof. The proof can be found by the enumeration of 128 representations. There is a slightly shorter proof which will be left to the reader. □

Lemma. *Let $n \in \mathbf{N}$, $n \geq 11$. Then every integer $x \in [129, \sigma_2(n!) - n!^2 - 129]$ can be represented as $x = \sum d_k^2$, where the d_k are different divisors of $n!$.*
Here $\sigma_m(x) = \sum_{d|x} d^m$.

Proof. Let $L_{kn} = [129, t]$ be the longest interval in $[129, \infty)$ whose integers can all be represented as a sum of different squares of some of the first k divisors of $n!$. Let $l_{kn} = |L_{kn}|$, the length of the interval. In the proof the notation will be abbreviate to $l_k = |L_k|$ if it is clear which n is meant.
In the second lemma we saw that $l_{10} = 128$. Notice that $l_{11} = 249\ (= 128 + 121)$. Any $x \leq 256$ is represented by the divisors lesser than or equal to 10, while the integers $257 \leq x \leq 377$ are represented using 11^2.
We will show in general that $l_{k+1} = l_k + d_{k+1}^2$ by induction, as long as $d_{k+1} < n!/2$. The proof will be given in two steps. In the first step we prove that $2d_{k+1}^2 < l_{k+1}$ given $2d_k^2 < l_k$ and $l_{k+1} = l_k + d_{k+1}^2$. In the second step we will prove that $l_{k+1} = l_k + d_{k+1}^2$ given $2d_k^2 < l_k$. Using these two steps and the basic assumption ($k = 11$) we can prove for arbitrary k that $l_{k+1} = l_k + d_{k+1}^2$.

First step
Given $2d_k^2 < l_k$ and $l_{k+1} = l_k + d_{k+1}^2$ we find that $2d_{k+1}^2 < l_{k+1}$.

Proof. The first lemma tells us that $d_{k+1}^2 < 2d_k^2$. Therefore we have $2d_{k+1}^2 < d_{k+1}^2 + 2d_k^2 < d_{k+1}^2 + l_k < l_{k+1}$. □

Second step
Given $2d_k^2 < l_k$ we find that $l_{k+1} = l_k + d_{k+1}^2$.

Proof. The proof is comparable to the proof above. □

For $x \in L_k$, it is clear that $x \in L_{k+1}$ as well, while for the numbers $x \in L_{k+1} \backslash L_k$ notice that $l_k + 128 < x \leq l_k + d_{k+1}^2 + 128$. If we use the number d_{k+1}^2 to represent the sum, we find for the rest $y = x - d_{k+1}^2$ that $l_k - d_{k+1}^2 + 128 < y \leq l_k + 128$. Using Lemma 1 again we have $l_k - d_{k+1}^2 + 128 > l_k - 2d_k^2 + 128 > 128$. Therefore $y \in L_k$.
We can rewrite the results $l_{k+1} = l_k + d_{k+1}^2$ as

$$l_k = \sum_{i=1}^{k} d_i^2 - 128,$$

for $k \leq m$, where $d_m = n!/3$.

In order to complete the proof of Lemma 3 we need to prove that $l_{(m+1)n} = l_{mn} + d^2_{n!/2}$, where $d_{m+1} = n!/2$. Notice that $\frac{1}{4} < \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \frac{1}{49}$. Therefore for arbitrary $x \in L_{(m+1)n}$, we find either $x \in L_{mn}$ or $x - \frac{1}{4}n!^2 \in L_{mn}$. Now we find

$$l_k = \sum_{i=1}^{k} d_i^2 - 128,$$

for $k \leq m+1$, where $d_{m+1} = n!/2$. This concludes the proof of this lemma. □

Theorem. *For every $q \in \mathbf{Q}$ such that $0 < q < \frac{\pi^2}{6} - 1$, a finite subset $K \in \mathbf{N}$ exists, such that*

$$\sum_{k \in K} k^{-2} = q$$

.

Proof. Let $q \in \mathbf{Q}$ with $0 < q < \frac{\pi^2}{6} - 1$. We can find an $n \in \mathbf{N}$ fulfilling each of the three following properties by choosing n sufficiently large. Moreover each of these properties is monotonic, meaning that if it is true for some n_0, it will be true for all $n > n_0$

- $n \geq 11$,
- If $q = a/b$, where a and b have no common divisors, then b divides $n!^2$,
- If $q = x/(n!)^2$, then n is chosen such that $128 < x < \sigma_2(n!) - (n!)^2 - 128$

To prove the existence of 3) notice that

$$\lim_{n \to \infty} \frac{\sigma_2(n!) - (n!)^2 - 128}{n!^2} = \frac{\pi^2}{6} - 1.$$

Now Lemma 3 may be applied, showing that x can be represented as sum of squares of different divisors of $n!$. This gives us the sought for representation of q. □

Remark

A solution of the Star Problem turns out to have been published in Ron Graham's 'On Finite Sums of Unit Fractions', *Proc. London Math. Soc.***(14)**, 1964, pp. 193–207. The basic ideas behind the two solutions are similar. Graham starts with a multiplicative set S, which in the Star Problem is the set of squares. Graham then defines $P(S)$, the set of sums of elements of S. Using the notation S^{-1} for the set of inverses of the elements of S, Graham shows that if $P(S)$ contains all positive integers, up to a finite number, $|S|$ is finite, and s_{n+1}/s_n is bounded, then $\frac{p}{q} \in P(S^{-1})$ whenever $q|s$ for some $s \in S$. Moreover, for every $\epsilon > 0$ there is an $s \in P(S^{-1})$ such that $s - \frac{p}{q} < \epsilon$.