Problemen/UWC

Universitaire Wiskunde Competitie

Edition 2005/1
For Session 2005/1 of the Universitaire Wiskunde Competitie we received submissions from Leendert Bleijenga; DESDA (Nijmegen); the team of Gerben Stavenga, Jan Kuipers, and Jaap Eldering; Hendrik Hubrechts; Ruud Jeurissen; and Jaap Spies.

Problem 2005/1-A Calculate

$$\sum_{n=1}^{\infty} \frac{1}{\sum_{i=1}^{n} i^2} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{1}{\sum_{i=1}^{n} i^3}.$$

Solution This problem has been solved by Leendert Bleijenga; DESDA (Nijmegen); the team of Gerben Stavenga, Jan Kuipers, and Jaap Eldering; Hendrik Hubrechts; Ruud Jeurissen; and Jaap Spies. The solution of Gerben Stavenga, et al. is given here.
Using the standard results $\sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1)$ and $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -\log 2$ we obtain:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{\sum_{i=1}^{n} i^2} &= \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{\frac{1}{6}n(n+1)(2n+1)} \\
&= \lim_{N\to\infty} 6\left(\sum_{n=1}^{N}\left(\frac{1}{n} + \frac{1}{n+1} - \frac{4}{2n+1}\right)\right) \\
&= \lim_{N\to\infty} 6\left(\sum_{n=1}^{N}\left(\frac{2}{n} - \frac{4}{2n+1}\right) - 1 + \frac{1}{N+1}\right) \\
&= \lim_{N\to\infty} 6\left(4\left(\sum_{n=1}^{N}\left(\frac{1}{2n} - \frac{1}{2n+1}\right)\right) - 1 + \frac{1}{N+1}\right) \\
&= \lim_{N\to\infty} 6\left(4\left(\sum_{n=1}^{2N+1}\frac{(-1)^n}{n} + 1\right) - 1 + \frac{1}{N+1}\right) \\
&= 6\left(-4\log 2 + 3\right).
\end{aligned}$$

Using the standard results $\sum_{i=1}^{n} i^3 = \frac{1}{4}n^2(n+1)^2$ and $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ we obtain:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{\sum_{i=1}^{n} i^3} &= \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{\frac{1}{4}n^2(n+1)^2} \\
&= \lim_{N\to\infty} 4\left(\sum_{n=1}^{N}\left(\frac{1}{n^2} + \frac{1}{(n+1)^2} - \frac{2}{n} + \frac{2}{n+1}\right)\right) \\
&= \lim_{N\to\infty} 4\left(\sum_{n=1}^{N}\frac{2}{n^2} - 1 + \frac{1}{(N+1)^2} - 2 + \frac{2}{N+1}\right) \\
&= \frac{4}{3}\left(\pi^2 - 9\right).
\end{aligned}$$

Problem 2005/1-B On a ruler of length 2 meter are placed 100 black ants and one red ant. Each ant walks with a speed of 1 meter/minute. If two ants meet then both turn 180^0. So does an ant that reaches the end of the ruler. At the start the red ant is exactly in the middle. Calculate the probability that the red ant is exactly in the middle after 4 minutes.

Solution This problem was solved by Michel Bel; DESDA (Nijmegen); the team of Gerben Stavenga, Jan Kuipers, and Jaap Eldering. We give Michel Bel's solution.
Let the ants take part in a relay race: when two ants meet, each ant gives his baton to the other ant. This way every baton moves in a single direction until the end of the ruler, after which it turns around and comes back. Hence: after four minutes every baton has

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made exactly one complete round and has come back to the same place. All batons are therefore at the same spot as at $t = 0$. (And the ants too...)
As the ants have not changed order every ant, including the red ant, has his own baton back. Conclusion: every ant is back at its own starting point. $P = 1$
In the original exercise one does not look after 4 minutes but after 2 minutes. That is interesting because the probability is no longer 1 and is not 0 either.

Problem 2005/1-C We call a triangle *integral* if the sides of the triangle are integral. Consider the integral triangles with rational circumradius.

1. Prove that for any positive integral p there are only a finitely many integral q such that there exists an integral triangle with circumradius equal to p/q.
2. Prove that for any positive integral q there exist infinitely many integral triangles with circumradius equal to p/q for an integral p with $\gcd(p,q) = 1$.

Solution This problem has been solved by the team of Gerben Stavenga, Jan Kuipers, and Jaap Eldering, and by Jaap Spies. We give the solution of the team of Gerben Stavenga et al.

Solution to 1
If the denominator q is bigger than $2p$, the diameter of the circumcircle is smaller than 1, so that no integral sides and also no integral triangles will fit in the circle. So the values for q, such that an integral triangle with circumradius equal to p/q possibly exists, are limited to $1 \dots 2p$, which is a finite set. □

Solution to 2
Take a denominator $q \geq 1$ and take

$$a = b = 4(q+1)(2q^2+2q+1) \quad \text{and} \quad c = 8(q+1)(2q+1).$$

It is clear that $a, b, c > 0$ and $a < b + c$, $b < a + c$ and $c < a + b$, so that there exists a integral triangle with sides equal to a, b and c.
The circumradius of a triangle is given by

$$R = \frac{abc}{\sqrt{(a+b+c)(a+b-c)(a+c-b)(b+c-a)}},$$

which in this case reduces to

$$R = \frac{(2q^2+2q+1)^2}{q}.$$

Since $gcd((2q^2+2q+1)^2, q) = 1$, there exists an integral triangle with circumradius p/q for an integral p with $gcd(p,q) = 1$, namely $p = (2q^2+2q+1)^2$. To obtain infinitely many integral triangles, we can scale this triangle with factors k, such that $gcd(k,q) = 1$, for example $k = (q+1)^n$ for $n \in \mathbf{N}$. □

Non-congruent triangles
It is also possible to construct an infinite set of non-congruent triangles:
The triangles constructed above are presented without derivation. We will give a sketchwise derivation now, and also show how they yield an infinite set of non-congruent integral triangles.
Instead of constructing integral triangles with rational circumradii, we will construct rational triangles with rational circumradii first. Scaling these will give integral triangles with rational circumradii. We will call its sides a, b and c and take $c = 1$.
The circumradius of a triangle is given by

$$R = \frac{c}{2\sin\gamma},$$

so we are looking for triangles with rational sines of the angles. For rational sides, the cosine formula,

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

gives that the cosines of the angles are rational. Furthermore since the sine and cosine are connected via $\sin^2\gamma + \cos^2\gamma = 1$, we take $\sin\gamma = p/r$ and $\cos\gamma = q/r$ with (p,q,r) a Pythagorean 3-tuple. These 3-tuples can be parametrised as follows

$$(m,n) \mapsto (m^2-n^2, 2mn, m^2+n^2),$$

so we take
$$\cos\gamma = \frac{m^2-n^2}{m^2+n^2} \quad \text{and} \quad \sin\gamma = \frac{2mn}{m^2+n^2},$$

for $m,n>0$ and integral.

Next we will try to find rational solutions (a,b) to the cosine formula with $c=1$ and fixed $\cos\gamma$:
$$a^2+b^2-2ab\cos\gamma = 1.$$

This equation can be rewritten as

$$(a\sin\gamma)^2 + (b-a\cos\gamma)^2 = 1,$$

so we are looking for rational point $(a',b') = (a\sin\gamma, b-a\cos\gamma)$ on the unit circle. These again are given by Pythagorean 3-tuples, which we will parametrise by $r,s>0$ integral:

$$a' = \frac{2rs}{r^2+s^2}. \quad \text{and} \quad b' = \frac{r^2-s^2}{r^2+s^2}$$

This gives as rational solutions (a,b,c) to the cosine formula with $\cos\gamma = \frac{m^2-n^2}{m^2+n^2}$:

$$a = \frac{(m^2+n^2)rs}{mn(r^2+s^2)}, \quad b = \frac{(m^2-n^2)rs + mn(r^2-s^2)}{mn(r^2+s^2)} \quad \text{and} \quad c=1$$

Scaling this triangle with a factor of $4mn(r^2+s^2)$ then gives the following integral solutions to the cosine formula with $\cos\gamma = \frac{m^2-n^2}{m^2+n^2}$:

$$a = 4(m^2+n^2)rs, \quad b = 4(m^2-n^2)rs + 4mn(r^2-s^2) \quad \text{and} \quad c = 4mn(r^2+s^2).$$

This integral triangle has circumradius equal to

$$R = (m^2+n^2)(r^2+s^2).$$

If we now take m,n,r and s such that q divides mn, q divides rs and $gcd(q,(m^2+n^2)(r^2+s^2)) = 1$, then the sides a,b,c are all divisible by q. Dividing them by q then results in an integral triangle with circumradius

$$R = \frac{(m^2+n^2)(r^2+s^2)}{q}.$$

In this way an infinite set of non-congruent integral triangles with rational circumradii with denominator q can be constructed.

The ones given above as solution to Part 2 of the problem are obtained by taking

$$m = q+1, \quad n = q, \quad r = q+1 \quad \text{and} \quad s = q,$$

and swapping b and c.

Problem 2005/1-D This problem has appeared earlier in Session 2004/2. It was reprinted in Session 2005/1 with a hint.

Quasiland has 30.045.015 inhabitants. Every two inhabitants are each others friend or foe. Any two friends have exactly one mutual friend and any two foes have at least ten mutual friends.

1. Describe the relations between the inhabitants.
2. Is it possible that less people live in Quasiland, while the inhabitants are still friend or foe as above?

Solution No submissions were received. We give the solution of Robbert Fokkink, who proposed the problem.

1) Note that $30.045.015 = \binom{30}{10}$. Let us start with 30 colours. Associate 10 colours to each inhabitant of Quasiland, so that no two different inhabitants have all colours alike. Two inhabitants are friends if their sets of associated colours are disjunct. For each pair of friends there is a third inhabitant who has been assigned the complimentary ten colours. This third inhabitant is unique. If the sets of two inhabitants are not disjunct, they are foes. In this case they have at least one colour in common, and together account for at most 19 colours. There are at least 11 complementary colours left. As there are at least 11 ways to choose 10 colours out of these, the two have at least 11 common friends.

2) If there are $4.686.825 = \binom{27}{9}$ inhabitants in Quasiland, the argument above still holds.