Problemen

| Problem Section

This Problem Section is open to everyone; everybody is encouraged to send in solutions and propose problems. Group contributions are welcome. We will select the most elegant solutions for publication. For this, solutions should be received before **15 Juli 2023**. The solutions of the problems in this issue will appear in one of the subsequent issues.

Problem A (proposed by Hendrik Lenstra)

Let R be a ring. We say $x \in R$ is *central* if $xy = yx$ for all $y \in R$. Suppose that for every $x \in R$ the element $x^2 - x$ is central in R. Show that R is commutative.

Problem B (proposed by Onno Berrevoets)

Isaac really likes apples, but does not like pears. He does not have any fruit now. Each time he visits

– Andrea, he gets 3 apples in exchange for 2 pears;
– Bob, he gets 3 pears;
– Caroline, he gets 1 apple and 1 pear.

Prove that the maximum number of apples Isaac can have after n visits equals $\lfloor 5n/9 \rfloor$.

Problem C (proposed by Daan van Gent)

For $S \subseteq \mathbb{Z}_{>0}$ write $\langle S \rangle$ for the submonoid of $\mathbb{Z}_{>0}$ generated by S. For $S \subseteq \mathbb{Z}_{>0}$ a *foundation* for S is a subset $C \subseteq \mathbb{Z}_{>0}$ for which $\langle C \rangle$ is minimal with respect to inclusion such that the elements of C are pairwise coprime and $S \subseteq \langle C \rangle$. For example, a foundation for $\{150, 180\}$ is $\{5, 6\}$.

a. Show that all subsets of $\mathbb{Z}_{>0}$ have a unique foundation.

Write $w(a,b)$ for the cardinality of the foundation for $\{a,b\}$ and let

$$f(n) = \min\{ab \mid a, b \in \mathbb{Z}_{>0}, w(a,b) = n\}.$$

b. Compute $f(11)$.
c.* What is the asymptotic behaviour of f?

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