Problemen

Problem Section

This Problem Section is open to everyone; everybody is encouraged to send in solutions and propose problems. Group contributions are welcome. We will select the most elegant solutions for publication. For this, solutions should be received before **15 July 2020**. The solutions of the problems in this issue will appear in the next issue.

Problem A (proposed by Hendrik Lenstra)

Let R be a ring, and write $R[[X,X^{-1}]]$ for the set of formal expressions $\sum_{i\in\mathbb{Z}} a_i X^i$ with all $a_i \in R$.

a. Suppose that $R[[X,X^{-1}]]$ has a ring structure with the following three properties.
   i. The sum is given by $\left(\sum_{i\in n\mathbb{Z}} a_i X^i\right) + \left(\sum_{i\in\mathbb{Z}} b_i X^i\right) = \sum_{i\in\mathbb{Z}} (a_i + b_i) X^i$,
   ii. For two formal power series in X, the product is the regular product of power series, and likewise for two formal power series in X^{-1}.
   iii. For $1 := X^0$, we have $X \cdot X^{-1} = 1$.

   Prove that R is the zero ring.
b. Prove that for every ring R, there exists a ring structure on $R[[X,X^{-1}]]$ satisfying properties I and II.

Problem B (proposed by Onno Berrevoets)

Let $n \geq 1$ be an integer, and let $p_1, \ldots, p_{n-1}$ be pairwise distinct prime numbers. Suppose that $(v_1, \ldots, v_n)^\top \in \mathbb{Z}^n$ is a non-trivial element of the kernel of

$$\begin{pmatrix} 1^{p_1} & 2^{p_1} & \ldots & n^{p_1} \\ 1^{p_2} & 2^{p_2} & \ldots & n^{p_2} \\ \vdots & \vdots & \ddots & \vdots \\ 1^{p_{n-1}} & 2^{p_{n-1}} & \ldots & n^{p_{n-1}} \end{pmatrix}.$$

Prove that

$$\max_k |v_k| \geq \frac{2}{n^2+n} \prod_{i=1}^{n-1} p_i.$$

Problem C (proposed by Onno Berrevoets)

Let $n, m, k \geq 2$ be positive integers. n students will attend a multiple-choice exam containing mk questions and each questions has k possible answers. A student passes the exam precisely when he/she answers at least $m+1$ questions correctly.

a. Suppose that $n = 2k$. Show that the students can coordinate their answers such that it is guaranteed that at least one student passes the exam.
b. Suppose that $n = 2k-1$. Does there exist a k for which the students can coordinate their answers such that it is guaranteed that at least one student passes the exam?

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