Problemen

| Problem Section

Problem A (folklore)

Denote for all positive rational numbers x by $f(x)$ the minimum number of 1's needed in a formula for x involving only ones, addition, subtraction, multiplication, division and parentheses. For example, $f(1)=1$, and $f(\frac{1}{3})=4$, as $\frac{1}{3}=\frac{1}{1+1+1}$ and as no such formula exists with at most three 1's. Note that $f(11)\neq 2$ (concatenation of ones is not allowed).
Moreover, denote for all positive rational numbers x by $h_2(x)$ the number $\log_2(p)+\log_2(q)$, where $\log_2$ denotes the base-2 logarithm, and where p, q are positive integers such that $x=\frac{p}{q}$ and $\gcd(p,q)=1$.
Show that for all x, we have

$$f(x)>\tfrac{1}{2}h_2(x).$$

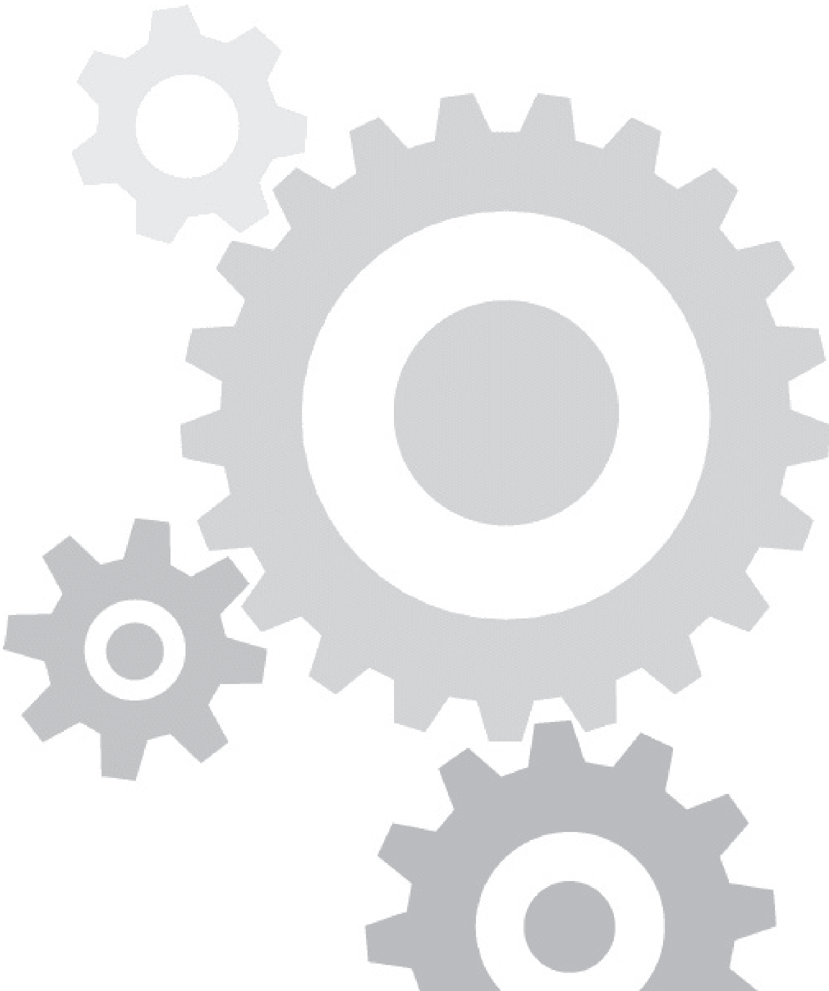
Problem B (folklore)

Suppose that there are $N\geq 2$ players, labeled $1,2,\dots,N$, and that each of them holds precisely $m\geq 1$ coins of value 1, m coins of (integer) value $n\geq 2$, m coins of value n^2, et cetera. A *transaction* from player i to player j consists of player i giving a finite number of his coins to player j. We say that an N-tuple $(a_1,a_2,\dots,a_N)$ of integers is (m,n)-*payable* if $\sum_{i=1}^N a_i=0$ and after a finite number of transactions, the i-th player has received (in value) a_i more than he has given away.
Show that for every N-tuple $(a_1,a_2,\dots,a_N)$ with $\sum_{i=1}^N a_i=0$ to be (m,n)-payable, it is necessary and sufficient that $m>n-\frac{n}{N}-1$.

Problem C (proposed by Wouter Zomervrucht)

For each integer $n\geq 1$ let c_n be the largest real number such that for any finite set of vectors $X\subset\mathbb{R}^n$ with $\sum_{v\in X}|v|\geq 1$ there exists a subset $Y\subseteq X$ with $|\sum_{v\in Y}v|\geq c_n$. Prove the recurrence relation

$$c_1=\tfrac{1}{2},\qquad c_{n+1}=\frac{1}{2\pi n c_n}.$$



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