Problemen

Problem Section

Problem A (folklore)

Determine

$$\sum_p \tfrac{1}{p} \prod_{q<p} (1 - \tfrac{1}{q}),$$

where p ranges over all prime numbers, and q ranges over all prime numbers less than p.

Problem B (proposed by Wouter Zomervrucht)

Let $\mathbb{N} = \{0, 1, \ldots\}$ denote the set of natural numbers, and let $\mathbb{N}^{2015}$ denote the set of 2015-tuples $\big(a(1), a(2), \ldots, a(2015)\big)$ of natural numbers. We equip $\mathbb{N}^{2015}$ with the partial order $\preceq$ for which $a \preceq b$ if and only if $a(k) \leq b(k)$ for all $k \in \{1, 2, \ldots, 2015\}$. We say that a sequence $a_1, a_2, \ldots$ in $\mathbb{N}^{2015}$ is *good* if for all $i < j$ we have $a_i \not\preceq a_j$.

– Show that all good sequences are finite.

We say that a sequence $a_1, a_2, \ldots$ in $\mathbb{N}^{2015}$ is *perfect* if it is good and for all i and for all $k \in \{1, 2, \ldots, 2015\}$ we have $a_i(k) \leq 2015i$.

– Does there exists a positive integer N such that all perfect sequences have length at most N?

Problem C (proposed by Marcel Roggeband)

The *(first) Bernoulli numbers* B_n for integers $n \geq 0$ are defined by the following recursive formula.

$$B_0 = 1,$$

$$B_n = -\sum_{i=0}^{n-1} \binom{n}{i} \frac{B_i}{n-i+1} \qquad \text{for } n > 0.$$

Show that the Bernoulli numbers satisfy the following identity for all $n > 1$:

$$B_n = n! \sum_{i=1}^{n-1} \sum_{\sigma} \frac{(-1)^{i-1}}{\sigma_1! \cdots \sigma_i!} (\tfrac{1}{2} - \tfrac{1}{\sigma_i+1}).$$

In this sum, σ runs through all i-tuples $(\sigma_1, \ldots, \sigma_i)$ of integers such that $\sigma_1 + \cdots + \sigma_i = n + i - 1$ and $\sigma_j \geq 2$ for all j.

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