# Problemen/UWC

## Universitaire Wiskunde Competitie

**Problem A** (Proposed by Jurjen Bos)
We are given a lamp and a sufficiently large number of synchronised time switches that can be turned on or off by the quarter of an hour and have a revolution time of 24 hours. We are going to mount a finite number of switches on top of each other, and put the lamp on top of the result. At the beginning, all time switches are synchronised at 24:00 hours. We define a *period* to be a time span in which the lamp is on for at least one quarter of an hour, and is off for at least one quarter of an hour, and which repeats itself. Which periods, shorter than 4 days, can be constructed?

**Problem B** (Proposed by Matthijs Coster)
Let $P = (0,0)$, $Q = (3,4)$. Find all points $T = (x, y)$ such that

- $x$ and $y$ are integers,
- the lengths of line segments $PT$ and $QT$ are integers.

**Problem C** (Proposed by Johan Bosman)
Let $n \geq 1$ be an integer and $f(x) = a_n x^n + \cdots + a_0$ be a polynomial with real coefficients. Suppose that $f$ satisfies the following condition:

$$|f(\xi)| \leq 1 \quad \text{for each } \xi \in [-1,1].$$

Consider the polynomial

$$g(x) = a_0 x^n + \cdots + a_n,$$

the reciprocal polynomial of $f$. Show that $g$ satisfies

$$|g(\xi)| \leq 2^{n-1} \quad \text{for each } \xi \in [-1,1].$$

**Problem D** (Proposed by Michiel Vermeulen)
This problem was proposed as Problem 2005/3-B(2). Since no solutions were submitted, the editors of the problem section have decided to reformulate this part.
Let $G$ be a group such that the maps $f_m, f_n : G \to G$ given by $f_m(x) = x^m$ and $f_n(x) = x^n$ are both homomorphisms.

- Show that $G$ is Abelian if $(m, n)$ is one of the pairs (4,11), (6,17).
- Show that there are infinitely many pairs $(m, n)$ such that $G$ is Abelian.
- Show that for every $m$ there are infinitely many $n$ such that $G$ is Abelian.
- Given a pair $(m, n)$, how are we able to predict whether $G$ is Abelian?

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