Problemen/UWC

Universitaire Wiskunde Competitie

Problem A (Unknown proposer)
We have $\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1$. Consequently partial sums must satisfy

$$\sum_{k \in K} \frac{1}{k(k+1)} < 1.$$

Show that for any $q \in \mathbf{Q}$ satisfying $0 < q < 1$, there exists a finite subset $K \subseteq \mathbf{N}$ so that

$$\sum_{k \in K} \frac{1}{k(k+1)} = q.$$

Problem B (Proposed by Matthijs Coster)
We consider the progressive arithmetic and geometric means of the function sequence $f_n(x) = x^{n-1}, n \in \mathbf{N}, x > 0, x \neq 1$. These are

$$A_n = A_n(x) = \frac{1}{n}(1 + x + x^2 + \cdots + x^{n-1}) = \frac{x^n - 1}{n(x-1)}$$

and

$$G_n = G_n(x) = (x^{1+2+\cdots+(n-1)})^{\frac{1}{n}} = x^{\frac{n-1}{2}}.$$

The *Martins-property* reads $A_{n+1}/A_n \geq G_{n+1}/G_n$. In our case this gives

$$\frac{n}{n+1} \frac{x^{n+1} - 1}{x^n - 1} \geq \sqrt{x}.$$

Prove, more generally, that

$$\frac{a}{a+1} \frac{x^{a+1} - 1}{x^a - 1} \geq \sqrt{x} \text{ for } a > -\frac{1}{2}, x > 0, x \neq 1.$$

Problem C (Proposed by Roger Hendrickx and Rob van der Waall)
A *finite geometry* is a geometric system that has only a finite number of points. For an *affine geometry*, the axioms are as follows:

1. Given any two distinct points, there is exactly one line that includes both points.
2. The parallel postulate: Given a line L and a point P not on L, there exists exactly one line through P that is parallel to L.
3. There exists a set of four points, no three collinear.

We denote the set of points by $\mathbf{P}$, and the set of lines by $\mathbf{L}$. Let σ be an automorphism of $(\mathbf{P}, \mathbf{L})$ (meaning that three collinear points of $\mathbf{P}$ are mapped onto three collinear points of $\mathbf{P}$ and three noncollinear points of $\mathbf{P}$ are mapped onto three noncollinear points of $\mathbf{P}$). Prove that there exists a point $P \in \mathbf{P}$ with $\sigma(P) = P$ or a line $L \in \mathbf{L}$ with $\sigma(L) = L$ or $\sigma(L) \cap L = \emptyset$.

Star Problem (Unknown proposer)
We have $\sum_{k=2}^{\infty} 1/k^2 = (\pi^2/6) - 1$. Consequently partial sums must satisfy

$$\sum_{k \in K} \frac{1}{k^2} < \frac{\pi^2}{6} - 1.$$

Given any $q \in \mathbf{Q}$ satisfying $0 < q < (\pi^2/6) - 1$, does there exist a finite subset $K \subseteq \mathbf{N} \setminus \{1\}$ so that

$$\sum_{k \in K} \frac{1}{k^2} = q?$$

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