Problemen/UWC

Universitaire Wiskunde Competitie

Problem A

Calculate

$$\sum_{n=1}^{\infty} \frac{1}{\sum_{i=1}^{n} i^2} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{\sum_{i=1}^{n} i^3}.$$

Problem B

On a ruler of length 2 meter are placed 100 black ants and one red ant. Each ant walks with a speed of 1 meter/minute. If two ants meet then both turn 180^0. So does an ant that reaches the end of the ruler. At the start the red ant is exactly in the middle. Calculate the probability that the red ant is exactly in the middle after 4 minutes.

Problem C

We call a triangle integral if the sides of the triangle are integral. Consider the integral triangles with rational circumradius.

1. Prove that for any positive integral p there are only a finitely many integral q such that there exists an integral triangle with circumradius equal to p/q.
2. Prove that for any positive integral q there exist infinitely many integral triangles with circumradius equal to p/q for an integral p with $\gcd(p, q) = 1$.

Problem D

This problem has appeared earlier in round 2004/2. At that time no submissions were received. We reprint it here with a hint.

Quasiland has 30.045.015 inhabitants. Every two inhabitants are each others friend or foe. Any two friends have exactly one mutual friend and any two foes have at least ten mutual friends.

1. Describe the relations between the inhabitants.
2. Is it possible that less people live in Quasiland, while the inhabitants are still friend or foe as above?

Problem E$\star$

Alice and Bob play a game. Alice places $n-1$ candles on a square cake. Bob places an extra candle in the bottom-left corner. Then, for each candle, he cuts a rectangular piece of cake such that the candle is at the bottom-left corner and no other candles are in the rectangle. Bob gets all these n pieces of cake.

1. Is it always possible for Bob to get more than half of the cake?
2. What is the optimal strategy for Alice to hold onto as much cake as possible?

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