Pierre Arnoux
Institut de Mathématiques de Marseille
Université d'Aix-Marseille et CNRS
pierre@pierrearnoux.fr

Maki Furukado
International Graduate School of Social Sciences
Yokohama National University
furukado-maki-pm@ynu.ac.jp

Mortal remains tangible memories of mathematicians

Shunji Ito's tombstone

The picture

The center of the tombstone shows a patch of a fractal tiling. A close examination shows six different tiles, highlighted in white in the middle. Above the patch, there are written three kanji, 伊藤家, read *Ito ke*, which means *Ito Family*.

Just below the patch, there is a map $1 \mapsto 34^{-1}$, $2 \mapsto 1$, $3 \mapsto 2$, $4 \mapsto 3$; on the bottom, we see a circle with two pairs of points, one pair inside the circle, one outside; on its right, an octagon divided in six parallelograms and a fractal domain, also divided in six pieces.

At the top of the tombstone, there is a series of six pictures: the first is the octagon seen at the bottom, and the next five are successive larger extensions of this octagon.

What does it mean?

The map above defines an endomorphism σ of the free group on four generators. It is an automorphism, with σ^{-1} given by $1 \mapsto 2$, $2 \mapsto 3$, $3 \mapsto 4$, $4 \mapsto 1^{-1}4$.

This map projects by abelianisation to a linear map of $\mathbb{R}^4$, with matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

The characteristic polynomial of A is $X^4 - X + 1$; it has no real root, and two pairs of conjugate complex roots, one pair of approximate value $-0.7271360 \pm 0.9340992i$ outside the unit circle, and the other one $0.7271360 \pm 0.4300142i$ inside the unit circle; these are the points depicted at the bottom left. Since all eigenvalues are distinct, there are two transverse invariant planes, one expanding and one contracting. The center bottom picture shows the projection π_e on the expanding plane, along the contracting plane, of the unit hypercube at the origin generated by the basis $(e_1, e_2, e_3, -e_4)$, or more exactly a union of six 2-faces with the same projection. One can see the origin point and the projections of the four projection of the canonical basis, which decrease geometrically in size from $\pi_e(e_1)$ to $\pi_e(e_4)$; the angle between two successive projection $\pi_e(e_k)$, $\pi_e(e_{k+1})$ is close to $128°$, the argument of the largest eigenvalue.

The last figure represents a fractal version of the unit hypercube, which we now explain.

The map σ acts on the free group, whose elements can be seen as reduced words on the letters $\{1,2,3,4,1^{-1},2^{-1},3^{-1},4^{-1}\}$, or as stepped lines in $\mathbb{R}^4$ along the basis vectors, and σ induces an extension of dimension 1, $E^1(\sigma)$, acting on these paths; for example, it sends the path along e_1 to a composite path along e_3, and then $-e_4$. A closed loop, like the boundary $121^{-1}2^{-1}$ of the square on e_1 and e_2, is sent to a closed loop, in that case $34^{-1}143^{-1}1^{-1}$, which is the boundary of the union of the square on e_3 and e_1 at the origin, and the square on $-e_4$ and e_1 at $-e_3$. In this way, one can build a extension of dimension 2, $E^2(\sigma)$, acting on squares. This gives a systematic way to build a substitution rule in higher dimension from a free group automorphism.

This is exactly what is depicted at the top: the action of $E^2(\sigma)$ on a 2-skeleton of the unit hypercube, or more exactly its projection on the expanding plane of σ. Since the unit hypercube is included in its image (it is a *seed*), on can show that we get in the limit an invariant stepped surface, which stays within bounded distance of the expanding plane, and projects to a polygonal quasi self-similar tiling. By renormalisation, it is easy to build an exactly self-similar tiling: this is what is shown in the main figure, and the projection of the renormalised seed is the fractal set shown at the bottom right.

One can find more details, a dual tiling, and an application to an explicit construction of a Markov partition for the toral automorphism associated with A in the paper [1]; the interest of this construction is that the automorphism considered here does not have the Pisot property, and the construction can be used in a quite general setting to obtain tiling substitutions.

Reference

1 Pierre Arnoux, Maki Furukado, Edmund Harriss and Shunji Ito, Algebraic numbers, free group automorphisms and substitutions on the plane, *Transactions of the American Mathematical Society* 363(9) (2011), 4651–4699.