Problem

| Problem Section

This Problem Section is open to everyone; everybody is encouraged to send in solutions and propose problems. Group contributions are welcome. We will select the most elegant solutions for publication. For this, solutions should be received before **15 October 2023**. The solutions of the problems in this issue will appear in one of the subsequent issues.

Problem A

Let $n>0$ be an integer and let $\varphi:\mathbb{R}^n \to \mathbb{R}^n$ be an isometry, i.e., a map such that for all $x,y \in \mathbb{R}^n$ we have $|\varphi(x)-\varphi(y)|=|x-y|$. Let $X \subset \mathbb{R}^n$ be a set such that $\{\varphi(x) \mid x \in X\} \subseteq X$. Show that if X is closed and bounded, then $\{\varphi(x) \mid x \in X\} = X$, and show that we can drop neither of these two assumptions.

Problem B

Let X be a normally distributed random variable and let $t \in \mathbb{R}_{>0}$. Show that $x \mapsto \mathbb{P}(X \le x+t | x \le X)$ is a decreasing function.

Problem C (proposed by Hendrik Lenstra)

Let $p=2n+1$ be a odd prime and consider the finitely presented group G with generators $x_1,\ldots,x_n$ and for each $0<i,j,k\le n$ such that $ij=k$ or $ij=p-k$ the relation $x_i x_j = x_k$. Show that G is a cyclic group of order n.

Edition 2023-2 We received correct solutions from Rik Biel, Rik Bos, Pieter de Groen, Alexander van Hoorn, Nicky Hekster, Marnix Klooster, Timo van der Laan, Thijmen Krebs, Kees Roos and Andrés Ventas.

Problem 2023-2/A (proposed by Hendrik Lenstra)

Let R be a ring. We say $x \in R$ is *central* if $xy=yx$ for all $y \in R$. Suppose that for every $x \in R$ the element x^2-x is central in R. Show that R is commutative.

Solution We received correct solutions from Rik Biel, Rik Bos, Alexander van Hoorn, Nicky Hekster, Thijmen Krebs, Kees Roos and Andrés Ventas. It was noted By Nicky Hekster that more general results have been given on sufficient conditions for a ring to be commutative. A survey is given by J. Pinter-Lucke, Commutativity conditions for rings: 1950–2005, *Expositiones Mathematicae* 25(2) (2007), 165–174.

For $x,y \in R$ we have that

$$xy+yx=(x+y)^2-x^2-y^2=[(x+y)^2-(x+y)]-[x^2-x]-[y^2-y]$$

is central. It follows that $x(xy+yx)=(xy+yx)x$, hence $x^2y=yx^2$. From $y(x^2-x)=(x^2-x)y$ it then follows that $xy=yx$. Hence R is commutative.

Problem 2023-2/B (proposed by Onno Berrevoets)

Isaac really likes apples, but does not like pears. He does not have any fruit now. Each time he visits

- Andrea, he gets 3 apples in exchange for 2 pears;
- Bob, he gets 3 pears;
- Caroline, he gets 1 apple and 1 pear.

Prove that the maximum number of apples Isaac can have after n visits equals $\lfloor 9n/5 \rfloor$.

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Solution We received correct solutions from Rik Biel, Pieter de Groen, Marnix Klooster and Thijmen Krebs. Partial solutions were received from Andrés Ventas and Timo van der Laan.

As many readers noticed, there was an error in the original problem statement: the maximum is $\lfloor 9n/5 \rfloor$ and not $\lfloor 5n/9 \rfloor$.

Value each pear as $3/5$ and each apple as 1. Then Andrea, Bob and Caroline give you value $9/5$, $9/5$ and $8/5$, respectively, hence after n visits the value of Isaac's fruit is at most $9n/5$. Therefore, he has at most $\lfloor 9n/5 \rfloor$ apples after n visits. For $n = 1,2,3,4,5$ he can also get this many apples, via C, BA, CCA, BCAA, BBAAA. The case $n = 0$ is trivial. Now let $n > 5$ be an integer. Let $r \geq 0$ be the remainder of n upon division by 5. We see inductively that Isaac can get $\lfloor 9r/5 \rfloor$ apples from r visits, and $\lfloor 9(n-r)/5 \rfloor = 9(n-r)/5$ apples from $n-r$ visits and thus

$$\lfloor 9r/5 \rfloor + 9(n-r)/5 = \lfloor 9n/5 \rfloor$$

apples after n visits.

Problem 2023-2/C (proposed by Daan van Gent)

For $S \subseteq \mathbb{Z}_{>0}$ write $\langle S \rangle$ for the submonoid of $\mathbb{Z}_{>0}$ generated by S. For $S \subseteq \mathbb{Z}_{>0}$ a *foundation* for S is a subset $C \subseteq \mathbb{Z}_{>1}$ for which $\langle C \rangle$ is minimal with respect to inclusion such that the elements of C are pairwise coprime and $S \subseteq \langle C \rangle$. For example, a foundation for $\{150, 180\}$ is $\{5, 6\}$.

a. Show that all subsets of $\mathbb{Z}_{>0}$ have a unique foundation.

Write $w(a,b)$ for the cardinality of the foundation for $\{a,b\}$ and let

$$f(n) = \min\{ab \mid a,b \in \mathbb{Z}_{>0}, w(a,b) = n\}.$$

b. Compute $f(11)$.
c.* What is the asymptotic behaviour of f?

Solution We received a correct solution from Thijmen Krebs. There was an error in the original problem statement. In the definition of a foundation it should be $C \subseteq \mathbb{Z}_{>1}$.

a. Let $S \subseteq \mathbb{Z}_{>1}$. Write $\mathcal{P}$ for the set of primes that divide any element of S and for primes p and $x \in \mathbb{Z}_{>0}$ write $v_p(x)$ for the exponent of p in x. This set induces an equivalence relation on $\mathcal{P}$ given by

$$p \sim q \Leftrightarrow (\forall s \in S)\ v_p(s)/g_p = v_q(s)/g_q$$

where $g_p = \gcd\{v_p(s) : s \in S\}$. We claim that $C = \{c_A : A \in \mathcal{P}/\sim\}$ with $c_A = \prod_{p \in A} p^{g_p}$ is the unique foundation of S.

For \(s\in S\) we have

$$s = \prod_{p \in \mathcal{P}} p^{v_p(s)} = \prod_{A \in (\mathcal{P}/\sim)} \prod_{p \in A} p^{(v_p(s)/g_p)\cdot g_p} = \prod_{[p] \in \mathcal{P}/\sim} c_{[p]}^{v_p(s)/g_p} \in \langle C \rangle,$$

so $S \subseteq \langle C \rangle$.

Let $M \subseteq \mathbb{Z}_{>0}$ be any submonoid generated by a set of pairwise coprime integers. We observe that for every $X \subseteq M$ the element

$$\operatorname{egcd} X = \prod_{\substack{p \text{ prime} \\ (\forall x \in X) p \mid x}} p^{\gcd\{v_p(x) : x \in X\}}$$

is also in M. As $c_{[p]} = \operatorname{egcd}\{s \in S : v_p(s) > 0\}$, we conclude that $\langle C \rangle$ is indeed minimal such that $S \subseteq \langle C \rangle$. Hence C is a foundation. With an argument similar to the proof of unique prime factorization one shows that if D and E are both foundations, and thus $\langle D \rangle = \langle E \rangle$, then $D = E$. Hence C is unique.

b. Let C be a foundation of $\{a,b\}$. We obtain

$$ab = \prod_{p \in \mathcal{P}} p^{v_p(a)+v_p(b)} = \prod_{\substack{(s,t) \in \mathbb{Z}^2_{\geq 0} \\ \gcd(s,t)=1}} \prod_{\substack{p \in \mathcal{P} \\ v_p(a) = g_p s \\ v_p(b) = g_p t}} p^{g_p(s+t)} = \prod_{\substack{(s,t) \in \mathbb{Z}^2_{\geq 0} \\ \gcd(s,t)=1}} c_{s,t}^{s+t},$$

where the $c_{s,t}$ make up C as in point a. If we order the factors by increasing 'weight' $s+t$ with ties broken arbitrarily, then the product is minimized, given that the factors are pairwise coprime, if the first n terms are the first n primes in decreasing order. With the exception of $w=1$, the number of pairs of weight w is $\varphi(w)$, while there are two pairs of weight 1. Hence

$$f(11) = 31 \cdot 29 \cdot 23^2 \cdot 19^3 \cdot 17^3 \cdot 13^4 \cdot 11^4 \cdot 7^5 \cdot 5^5 \cdot 3^5 \cdot 2^5.$$

c. We will omit out some steps for brevity. Let

$$g(n) = \frac{2\pi}{\sqrt{27}} n\sqrt{n}\log n.$$

We claim that $\log f(n) \sim g(n)$.

Order the index set $\{x_1, x_2, \ldots\}$ of the product as before and let $w_i = s+t$ be the weight of $x_i = (s,t)$. Write $\sigma(n) = \sum_{k=1}^{n} \varphi(n)$ and note that $\sigma(w_i) \sim i$. By a result of A. Walfisz we have $\sigma(k) \sim 3k^2/\pi^2$. Hence $w_i \sim \pi\sqrt{i/3}$. With p_i the i-th prime we have $p_i \sim i \log i$. We split the sum

$$\log f(n) = \sum_{i=1}^{n} w_i \log p_{n+1-i}$$

in three parts with indices $[1,N]$, $(N, n-N)$ and $(n-N, n)$ with $N = n/\log(n)$. The first and last part we estimate by

$$\leq N w_n \log p_n \sim \frac{n}{\log n}(\pi\sqrt{n/3})(n\log n) = o(g(n)).$$

For the remaining sum we have

$$\begin{aligned}
&\sim \sum_{i=N}^{n-N} \pi\sqrt{i/3}\,(\log(n+1-i) + \log\log(n+1-i)) \\
&= (1+o(1))\frac{\pi}{\sqrt{3}} \sum_{i=N}^{n-N} \sqrt{i}\log(n+1-i) \\
&= (1+o(1))\frac{\pi}{\sqrt{3}} \log(n) \cdot \tfrac{2}{3} n^{3/2} \\
&= (1+o(1))\, g(n).
\end{aligned}$$

Rectifications: From Thijmen Krebs correct solutions were also received for 2021-3B and 2021-3C (as well as a correct solution for 2021-3A). For 2019-3B it was noted that this problem corresponded to 2016-1C.