Problemen

| Problem Section

This Problem Section is open to everyone; everybody is encouraged to send in solutions and propose problems. Group contributions are welcome. We will select the most elegant solutions for publication. For this, solutions should be received before **15 Juli 2023**. The solutions of the problems in this issue will appear in one of the subsequent issues.

Problem A (proposed by Hendrik Lenstra)

Let R be a ring. We say $x \in R$ is *central* if $xy = yx$ for all $y \in R$. Suppose that for every $x \in R$ the element $x^2 - x$ is central in R. Show that R is commutative.

Problem B (proposed by Onno Berrevoets)

Isaac really likes apples, but does not like pears. He does not have any fruit now. Each time he visits

– Andrea, he gets 3 apples in exchange for 2 pears;
– Bob, he gets 3 pears;
– Caroline, he gets 1 apple and 1 pear.

Prove that the maximum number of apples Isaac can have after n visits equals $\lfloor 5n/9 \rfloor$.

Problem C (proposed by Daan van Gent)

For $S \subseteq \mathbb{Z}_{>0}$ write $\langle S \rangle$ for the submonoid of $\mathbb{Z}_{>0}$ generated by S. For $S \subseteq \mathbb{Z}_{>0}$ a *foundation* for S is a subset $C \subseteq \mathbb{Z}_{>0}$ for which $\langle C \rangle$ is minimal with respect to inclusion such that the elements of C are pairwise coprime and $S \subseteq \langle C \rangle$. For example, a foundation for $\{150, 180\}$ is $\{5, 6\}$.

a. Show that all subsets of $\mathbb{Z}_{>0}$ have a unique foundation.

Write $w(a,b)$ for the cardinality of the foundation for $\{a,b\}$ and let

$$f(n) = \min\{ab \mid a, b \in \mathbb{Z}_{>0}, w(a,b) = n\}.$$

b. Compute $f(11)$.
c.* What is the asymptotic behaviour of f?

Edition 2023-1 We received solutions from Michel Bel, Carsten Dietzel and Albert Visser.

Problem 2023-1/A

Does there exist a partitioning X of $\mathbb{R}$ into infinite sets such that for every *choice map* $c: X \to \mathbb{R}$, i.e. a map c such that $c(S) \in S$ for all $S \in X$, the image of c is dense in $\mathbb{R}$?

Solution We received correct solutions from Michel Bel, Carsten Dietzel and Albert Visser. We show here the solution from Albert Visser: Let's say that the triple (m,n,p) is *adequate* if m is an odd integer, n is a natural number, and p is an odd prime. We define

$$A_{m,n,p} = \left\{ \frac{m}{2^n} + \frac{1}{p^k} \;\middle|\; k \in \mathbb{Z}_{\geq 0} \right\} \quad \text{where } (m,n,p) \text{ is adequate},$$

and write A_∞ for the complement in $\mathbb{R}$ of the union of all such $A_{m,n,p}$. Consider $\mathcal{X} = \{A_{m,n,p} \mid (m,n,p) \text{ is adequate}\} \cup \{A_\infty\}$. Clearly, all elements of $\mathcal{X}$ are infinite and the union of $\mathcal{X}$ is $\mathbb{R}$. We show that the elements of $\mathcal{X}$ are pairwise disjoint. By definition, A_∞ is disjoint from the $A_{m,n,p}$. We note that for adequate (m,n,p) and $k \in \mathbb{Z}_{\geq 0}$ the fraction

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$$\frac{m}{2^n} + \frac{1}{p^k} = \frac{mp^k + 2^n}{2^n p^k},$$

cannot be further simplified. This means that, if

$$\frac{2^{n_1} + m_1 p_1^{k_1}}{2^{n_1} p_1^{k_1}} = \frac{2^{n_2} + m_2 p_2^{k_2}}{2^{n_2} p_2^{k_2}}$$

for adequate (m_i, n_i, p_i), we deduce from the denominators that $n_1 = n_2$, $p_1 = p_2$ and $k_1 = k_2$, and finally $m_1 = m_2$. Hence the $A_{m,n,p}$ are pairwise disjoint.

Finally, consider any real r and any $\delta > 0$. We can find $m \in \mathbb{Z}$ and $n \in \mathbb{Z}_{\geq 0}$ such that $|r - m2^{-n}| < \delta/2$ by considering the binary expansion of r, and an odd prime p with $1/p < \delta/2$. It follows that for all $q \in A_{m,n,p}$ we have $|r - q| < \delta$. In particular, this holds for $q = c(A_{m,n,p})$, so the image of c is dense.

Problem 2023-1/B

Show that for all $k \in \mathbb{Z}$ there exists an $x \in \mathbb{Q}$ for which there are at least two subsets $S \subseteq \mathbb{Z}_{\geq 1}$ such that $\sum_{s \in S} s^k = x$.

Solution We received a correct solution from Carsten Dietzel: First assume that $k \geq 0$. Let n be an integer with $2^n > n^{k+1} + 1$. Denote by P_n the power set of $\{1, \ldots, n\}$ and by $\mathbb{N}$ the set of positive integers. Define the map

$$\begin{aligned} \sigma : P_n &\to \mathbb{N}, \\ S &\mapsto \sum_{s \in S} s^k. \end{aligned}$$

For $S \in P_n$ we have $\sigma(S) \leq n \cdot n^k = n^{k+1}$. Therefore, the range of σ is a subset of $\{0, \ldots, n^{k+1}\}$. By the pigeonhole principle, σ is not injective, i.e., there are distinct $S_1, S_2 \in P_n$ with $\sigma(S_1) = \sigma(S_2)$.

Now let $k < 0$. From the case $k \geq 0$ we know that there are finite subsets $T_1, T_2 \subset \mathbb{N}$ such that $\sum_{t \in T_1} t^{-k} = \sum_{t \in T_2} t^{-k}$. Let N be the least common multiple of the elements of $T_1 \cap T_2$. Define $S_i = \{N/t \mid t \in T_i\}$ for $i = 1, 2$. Then for $i = 1, 2$ we have

$$\sum_{s \in S_i} s^k = \sum_{t \in T_i} (t/N)^{-k} = N^k \sum_{t \in T_i} t^{-k}$$

and thus $\sum_{s \in S_1} s^k = \sum_{s \in S_2} s^k$. Hence, S_1 and S_2 are as desired!

Problem 2023-1/C (proposed by Daan van Gent)

For a group G and $g \in G$ write $c(g) = \{hgh^{-1} \mid h \in G\}$ and $G^\circ = \{g \in G \mid \#c(g) < \infty\}$.

a. Show that G° is a normal subgroup of G and that $G^{\circ\circ} = G^\circ$.

b. Now define $G_\circ = G/G^\circ$. Show that there exists a group G for which the sequence $G, G_\circ, G_{\circ\circ}, \ldots$ does not stabilize, i.e. for none of the groups H in the sequence we have $H^\circ = 1$.

Solution We received a correct solution from Carsten Dietzel. The solution shown here is by Daan van Gent.

a. For all $g, h \in G$ we have $c(g^{-1}) = \{a^{-1} \mid a \in c(g)\}$ and $c(gh) \subseteq \{ab \mid a \in c(g), b \in c(h)\}$. Thus for $g, h \in G^\circ$ both $c(g^{-1})$ and $c(gh)$ are finite and $c(1) = \{1\}$, so G° is a subgroup of G. Then note that $c(g) = c(h)$ for conjugates g and h, so G° is normal.

Clearly $G^{\circ\circ} \subseteq G^\circ$. Suppose $g \in G^\circ$. Since g has only finitely many conjugates in G, it certainly has only finitely many conjugates in a subgroup, so $g \in G^{\circ\circ}$.

b. For a family of groups $(G_i)_{i \in I}$ we define the *restricted product*

$$\bigoplus_{i \in I} G_i = \left\{ (x_i)_{i \in I} \in \prod_{i \in I} G_i \mid x_i = 1 \text{ for all but finitely } i \right\}.$$

We inductively define $P_0 = \{1\}$ and for $k \geq 0$ define

$$F_k = \bigoplus_{\substack{H \triangleleft P_k \\ \#(P_k/H) < \infty}} \mathbb{Z}_2^{P_k/H} \quad \text{and} \quad P_{k+1} = F_k \rtimes P_k,$$

where $\mathbb{Z}_2$ is the group of 2-adic integers and $\pi \in P_k$ acts on each factor $\mathbb{Z}_2^{P_k/H}$ as $(\varphi_{H,x})_{x \in P_k/H} \mapsto (\varphi_{H,\pi x})_{x \in P_k/H}$.

Note that every $x \in F_k$ has a finite orbit under P_k, because all P_k/H are finite and each $x \in F_k$ has finite support. Hence $F_k \subseteq P_{k+1}^{\circ}$. Conversely, suppose $(\varphi,\pi) \in P_{k+1}^{\circ}$. For $(\psi,1) \in P_{k+1}$ we have $(\psi,1)(\varphi,\pi)(\psi,1)^{-1} = ((\psi_{H,x} + \varphi_{H,x} - \psi_{H,\pi x})_{H,x}, \pi)$. If $\pi x \neq x$ for some (H,x), then with $\psi_{H,x} = 0$ and $\psi_{H,\pi x}$ ranging over $\mathbb{Z}_2$ we obtain infinitely many conjugates of (φ,π). Hence $\pi x = x$, and π acts trivially on every finite quotient of P_k. One shows inductively that P_k is profinite, hence $\pi = 1$. Thus $P_{k+1}^{\circ} = F_k$ and $(P_{k+1})_{\circ} = P_k$.

Finally, let $G = \bigoplus_{k=0}^{\infty} P_k$. Observe that $G^{\circ} = \bigoplus_{k=0}^{\infty} (P_k^{\circ}) = \bigoplus_{k=1}^{\infty} F_{k-1} \neq 1$, while $G_{\circ} \cong G$. Hence the sequence never stabilizes.