Problemen

| Problem Section

This Problem Section is open to everyone; everybody is encouraged to send in solutions and propose problems. Group contributions are welcome. We will select the most elegant solutions for publication. For this, solutions should be received before **15 April 2022**. The solutions of the problems in this issue will appear in one of the subsequent issues.

Problem A (proposed by Hendrik Lenstra)

Let R be a ring. We say $x \in R$ is a *unit* if there exists some $y \in R$ such that $xy = yx = 1$ and write R^* for the set of units of R. Show that $1 < \#(R \setminus R^*) < \infty$ implies $1 < \#R < \infty$.

Problem B (proposed by Hendrik Lenstra)

Let G be a group. For $n \in \mathbb{Z}_{>0}$ write $G[n] = \{g \in G \mid g^n = 1\}$ and $G^n = \{g^n \mid g \in G\}$.

1. Suppose G is abelian and $m, n \in \mathbb{Z}_{>0}$. Show that $G[n] \subseteq G^m$ if and only if $G[m] \subseteq G^n$.
2. Show that there exist $m, n \in \mathbb{Z}_{>0}$ such that the above is false when we drop the assumption that G is abelian.

Problem C (proposed by Onno Berrevoets)

Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function. Suppose that $a < b < c$ are real numbers such that $f(a) = f(b) = f(c) = 0$. Prove that there exists $x \in (a, c)$ such that

$$f'(x) + f''(x) = f(x)^2 + 2f(x)f'(x).$$

Edition 2021-2 We received solutions from Rik Biel, Rob Eggermont and Sander Scholtus. The solution of Problem B will appear in a future issue.

Problem 2021-2/A (proposed by Onno Berrevoets)

A hot frying pan contains 2^{2021} potato slices. Each time we toss the slices, each slice has a chance of 0.5 to land on its other side. These probabilities are individually independent. How often do we need to toss the slices so that with probability at least 0.5 all slices will have lain on both sides?

Solution This solution is submitted by Sander Scholtus (also solved by Rik Biel). The answer is: 2022 times. Denote the probability that a potato slice has landed on both sides after tossing k times by p_k. It is clear that $p_0 = 0$ and $p_1 = \frac{1}{2}$. Generally, p_k equals 1 minus the probability that a potato slice has not flipped after tossing k times, i.e., we have $p_k = 1 - (1 - p_1)^k = 1 - 2^{-k}$ for all $k \in \{0, 1, 2, \dots\}$. By independence of the potato slices, we now find that all $N := 2^{2021}$ slices are flipped after tossing k times equals $p_k^N = (1 - 2^{-k})^N$. Hence, we are asked to find the minimal value of k for which we have

$$\left(1 - \frac{1}{2^k}\right)^N > \frac{1}{2}.$$

The left hand side is clearly a strictly increasing function of k. Now consider $k = 2021$. In this case, we have

$$\left(1 - \frac{1}{2^k}\right)^N = \left(1 - \frac{1}{N}\right)^N.$$

As N is very large, we can approximate this number sufficiently well with $1/e < \frac{1}{2}$ and

Redactie: Onno Berrevoets en Daan van Gent
problems@nieuwarchief.nl
www.nieuwarchief.nl/problems

conclude that $k=2021$ is too small to achieve our desired result. We try the next value $k=2022$. In this case we find

$$\left(1-\frac{1}{2^k}\right)^N=\left(1-\frac{1}{2N}\right)^N=\sqrt{\left(1-\frac{1}{N}\right)^N}\approx\sqrt{\frac{1}{e}}>\frac{1}{2}.$$

Hence $k=2022$ is the desired number to toss the frying pan.

Justifications of the approximations in the previous argument can be found in the following inequalities:

$$\left(1-\frac{1}{x}\right)^x<\frac{1}{e}<\left(1-\frac{1}{x}\right)^{x-1}.$$

Indeed, in the limit $x\to\infty$, these inequalities become equalities. Moreover, $(1-\frac{1}{x})^x$ and $(1-\frac{1}{x})^{x-1}$ are respectively increasing and decreasing functions, which can be shown by taking the derivative and using the Mercator series of the logarithm.

Problem 2021-2/C (proposed by Onno Berrevoets)

Let R be a commutative ring, and consider the set X of R-ideals J with $J^2\neq J$. Suppose that I is a maximal element of X (with respect to inclusion). Prove that I is a maximal ideal of R.

Solution This solution is submitted by Rob Eggermont. Without loss of generality, we may assume $I^2=0$. To prove I is maximal, it suffices to prove that for all $x\in R\setminus I$, x is invertible $(\mathrm{mod}\, I)$. Let $x\in R\setminus I$. By assumption of maximality, $(xR+I)^2=xR+I$, so there are $r\in R$, $i\in I$ such that $x=rx^2+ix$. Our aim is to show $rx=1(\mathrm{mod}\, I)$. Write $u=1-rx$. Note that we may rewrite $x=rx^2+ix$ as $xu=ix$. We claim that $u^3=u^2$. Indeed, we have

$$u^3=u^2(1-rx)=u^2-rxu^2=u^2-rxi^2=u^2.$$

from which it follows that $u^2(1-u)=0$. As a consequence, we see that $(1-u)R\cap u^2R=0$: for any $r,s\in R$ with $(1-u)s=u^2r$, we have $u^2r=u^4r=u^2(1-u)s=0$.

Consider the ideal J generated by u^2 and I. Observe that $J^2=(u^2R+I)^2=u^4R+u^2I=u^2R$. If $u^2\notin I$, then $J^2=J$ by our assumption on maximality, so then J equals u^2R, hence $I\subseteq u^2R$. In particular, we have $(1-u)I=0$ since $(1-u)u^2=0$. Consider the ideal J' generated by $(1-u)$ and I. We have $J'^2\subseteq(1-u)R$. Recall that $(1-u)R\cap u^2R=0$. Since I is not zero and contained in u^2R, it follows that J'^2 is strictly contained in J', hence $J'=I$, so $1-u\in I\subseteq u^2R$. From $(1-u)R\cap u^2R=0$ it now follows that $1-u=0$, so $u=1$. However, this means $rx=0$, and we find $x=ix$, contradicting our assumption that x is not in I. So we must have $u^2\in I$. This yields $u^2=u^4=0$.

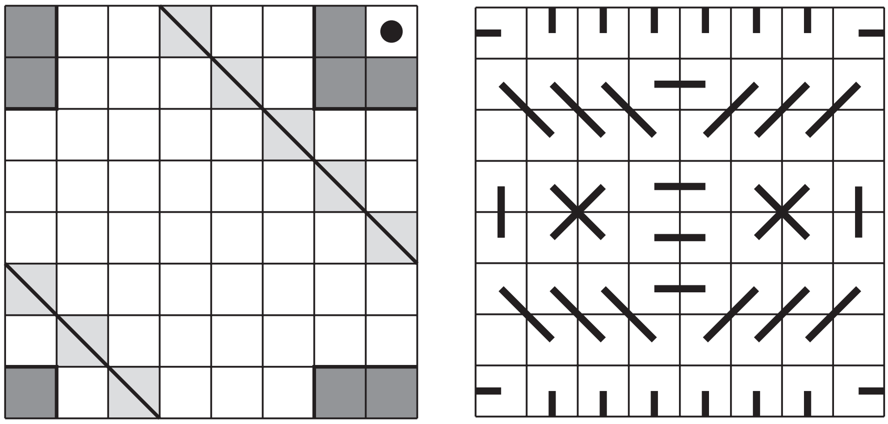
It is easy to see that $(uR+I)^3=0$, since $I^2=0$ and $u^2=0$. Since $uR+I$ is not zero, it follows that $(uR+I)^2$ is strictly contained in $uR+I$. Hence $u\in I$, and therefore $rx=1(\mathrm{mod}\, I)$. So x is invertible $(\mathrm{mod}\, I)$, as was to be shown.

Edition 2021-3 We received solutions from Rik Biel, Mike Daas, Thijmen Krebs and Wim Nuij.

Problem 2021-3/A (proposed by Daan van Gent)

Write $T=(\mathbb{Z}/8\mathbb{Z})^2$ for the *torus chessboard*. For every square $t\in T$ its *neighbours* are the squares in the set $\{t+d\mid d\in D\}$ for $D=\{-1,0,1\}^2\setminus\{(0,0)\}$. A *line* is a set of squares of the form $\{t+nd\mid n\in\mathbb{Z}\}$ for $t\in T$ and $d\in D$. The left figure on the next page gives an example of the neighbours (coloured dark grey) of the dotted square, as well as an example of a line (coloured light grey). Disprove or give an example: There exists a pairing of neighbouring squares, i.e. a partitioning of T into sets $\{s,t\}$ of size 2 such that s and t are neighbours, such that every line contains a pair.

Solution The right figure shows the solution by Thijmen Krebs (also solved by Wim Nuij):



Problem 2021-3/B (proposed by Hendrik Lenstra)

Write φ for the Euler totient function. Determine all infinite sequences $(a_n)_{n\in\mathbb{Z}_{\geq 0}}$ of positive integers satisfying $\varphi(a_{n+1}) = a_n$ for all $n \geq 0$.

Solution Let $(a_n)_{n\geq 0}$ satisfy $\varphi(a_{n+1}) = a_n$ for all $n \geq 0$. Write $X = \{2^a \cdot 3^b \mid a,b \in \mathbb{Z}_{\geq 0}\}$.
Claim: $(a_n)_n \subset X$.
Proof: We will prove this claim by contradiction, so assume that there exists an integer $N \geq 0$ for which $a_N \notin X$. Then for all $n \geq N$ we have that $a_n \notin X$, and without loss of generality $N = 0$. Then the sequence $(\mathrm{ord}_2(a_n))_{n\geq 0}$ is weakly decreasing. Therefore, without loss of generality we may assume that the sequence $(\mathrm{ord}_2(a_n))_{n\geq 0}$ is constant. Now by dividing the sequence $(a_n)_n$ by $2^{\mathrm{ord}(a_0)-1}$ we may assume that $\mathrm{ord}_2(a_n) = 1$ for all n. For every n, write $a_n = 2b_n$ with $b_n > 1$ odd. Then for every $n \geq 1$ we have that b_n is a prime power: otherwise $\mathrm{ord}_2(a_{n-1}) = \mathrm{ord}_2(\varphi(b_n))$ would be too large. Hence write $b_n = p_n^{c_n}$ with p_n an odd prime and $c_n \in \mathbb{Z}_{\geq 1}$. Then for every $n \geq 2$ we see that $\varphi(b_n)/2 = p_n^{c_n-1} \cdot (p_n - 1)/2$ is a prime power, and since $p_n \neq 3$ we find that $c_n = 1$. Hence, $(b_n)_{n\geq 2}$ is a sequence of prime numbers satisfying $b_n = 2b_{n-1} + 1$. Then for all $n \geq 0$ we find that $b_{n+2} = 2^n(b_2 + 1) - 1$. Let p be any prime divisor of b_2. Then p divides b_{n+2} whenever $2^n \equiv 1$ modulo p, so in this case $b_{n+2} = p$. This happens infinitely many times and is a contradiction since $(b_n)_n$ is strictly increasing. □

By the previous claim we conclude that for all $n \geq 0$ we have $a_n \in X$. The sequences $(a_n)_n \subset X$ that suffice the relation $\varphi(a_{n+1}) = a_n$ for all $n \geq 0$ are precisely those $(a_n)_n$ for which $(a_{n+1}/a_n)_n$ is a weakly increasing sequence in $\{2,3\}$.

Problem 2021-3/C (proposed by Hendrik Lenstra)

Let R be a ring. We say $x \in R$ is a *unit* if there exists some $y \in R$ such that $xy = yx = 1$, and $x \in R$ is *idempotent* if $x^2 = x$. Show that if every unit of R is central, then every idempotent of R is central.

Solution This solution is submitted by Mike Daas (also solved by Rik Biel). Assume that every unit of R is central. Note that for every idempotent $e \in R$ it holds that $e(1-e) = (1-e)e = 0$.
Claim: $e \in R$ is a central idempotent if and only if $(1-e)Re = eR(1-e) = 0$.
Proof. If e is a central idempotent, then $(1-e)Re = (1-e)eR = 0$ and similarly $eR(1-e) = 0$. Now suppose $(1-e)Re = eR(1-e) = 0$. For any $x \in R$ we have $0 = (1-e)xe = xe - exe$ so $xe = exe$ and similarly $ex = exe$. Hence $xe = ex$ and e is central. Taking $x = 1$ in the above shows $e = e^2$, so e is idempotent. □

Let $e \in R$ be idempotent. Suppose $x \in (1-e)Re$. Then $x^2 = 0$, so $1 + x$ has inverse $1 - x$. By assumption $1 + x$ is central. As 1 is trivially central we conclude that x is central. Then $x = (1-e)xe = (1-e)ex = 0$, so $(1-e)Re = 0$. Similarly $eR(1-e) = 0$, so e is central by the above.