This Problem Section is open to everyone; everybody is encouraged to send in solutions and propose problems. Group contributions are welcome. We will select the most elegant solutions for publication. For this, solutions should be received before **15 April 2021**. The solutions of the problems in this issue will appear in the next issue.

**Problem A** (proposed by Daan van Gent and Hendrik Lenstra)

Let $G$ and $A$ be groups, where $G$ is denoted multiplicatively and where $A$ is abelian and denoted additively. Assume that $A$ is 2-torsion-free, i.e. it contains no element of order 2.

Suppose that $q: G \to A$ is a map satisfying the parallelogram identity: for all $x, y \in G$ we have

$$q(xy) + q(xy^{-1}) = 2q(x) + 2q(y).$$

Prove that for all $x, y \in G$ we have $q(xy^{-1}y^{-1}) = 0$.

**Problem B** (folklore)

Prove that every Jordan curve (i.e. every non-self-intersecting continuous loop in the plane) contains four points $A, B, C, D$ such that $ABCD$ forms a rhombus.

**Problem C** (proposed by Daan van Gent)

A **directed binary graph** is a finite vertex set $V$ together with maps $e_1, e_2 : V \to V$. (The edges are formed by the ordered pairs $(v, e_i(v))$ with $i \in \{1, 2\}$.)

For $a, b, c, d \in \mathbb{Z}_{>0}$, an $(a:b)$-to-$4(c:d)$ **distributive graph** is a directed binary graph $G$ together with distinct vertices $s, t_1, t_2 \in V$ such that $G$ interpreted as a Markov chain has the following properties:

1. For all $v \in V$ the edges $(v, e_1(v))$ have transition probability $\frac{1}{a+b}$ and edges $(v, e_2(v))$ have probability $\frac{1}{a+b}$.
2. It has the initial state $s$ with probability 1.
3. Both $t_1$ and $t_2$ connect to themselves, meaning $e_1(t_i) = t_i$ for all $i, j \in \{1, 2\}$.
4. It has a unique stationary distribution of $t_1$ with probability $\frac{1}{a+c}$ and $t_2$ with probability $\frac{1}{a+d}$.

Show that for all $a, b, c, d \in \mathbb{Z}_{>0}$ there exists an $(a:b)$-to-$4(c:d)$ distributive graph.

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**Edition 2020-3** We received solutions from Brian Gilding, Pieter de Groen, Marco Pouw and Ludo Pulles.

**Problem 2020-3/A** (proposed by Onno Berrevoets)

Let $f : (-1.1) \to \mathbb{R}$ be a function of class $C^\infty$, i.e., all higher derivatives of $f$ exist on $(-1.1)$. Let $c \geq 0$ be a real number. Suppose that for all $x \in (-1.1)$ and all $n \in \mathbb{Z}_{\geq 0}$ we have $f^{(n)}(x) \geq -c$. Also assume that for all $x \in (-1, 0]$ we have $f(x) = 0$. Prove that $f$ is the zero function.

**Solution** We received solutions from Brian Gilding, Pieter de Groen and Marco Pouw. This solution is based on the one by Brian Gilding, who not only gives a very concise solution, but also shows that some of the assumptions can be weakened.

Since $f \in C^\infty(-1.1)$ and $f \equiv 0$ in $(-1.0)$, $f^{(n)}(0) = 0$ for every $n \in \mathbb{Z}_{\geq 0}$. Consequently, for arbitrary $x \in (0, 1)$ and $n \in \mathbb{Z}_{\geq 2}$, Taylor's Theorem (or repeated integration by parts, following the proof by Pieter de Groen) gives
\[ f(x) = \int_0^x \frac{(x-t)^{n-1}}{(n-1)!} f^{(n)}(t) dt \geq -c \int_0^x \frac{(x-t)^{n-1}}{(n-1)!} dt = -\frac{c x^n}{n!}. \]

Likewise,
\[ f'(x) = \int_0^x \frac{(x-t)^{n-2}}{(n-2)!} f^{(n)}(t) dt, \]
and this gives us
\[ f(x) = \frac{x}{n-1} f'(x) = -\int_0^x \frac{t(x-t)^{n-2}}{(n-1)!} f^{(n)}(t) dt \geq -c \int_0^x \frac{t(x-t)^{n-2}}{(n-1)!} dt = \frac{c x^n}{n-1!}. \]

Passage to the limit \( n \to \infty \) yields \( f(x) = 0 \) for all such \( x \).

The assumption \( f^{(n)} \geq -c \) in \((-1,1)\) for every \( n \in \mathbb{Z}_{\geq 0} \) for some nonnegative real number \( c \) can be relaxed to \( \pm f^{(n)} \leq n! g_n \) for every \( n \in \mathbb{Z}_{\geq 0} \) for a sequence of nonnegative functions \( \{g_n; n \in \mathbb{Z}_{\geq 0}\} \subset L_{\mathbb{R}}(\mathbb{R}) \) with the property \( x^n g_n \mathcal{L}(x) \to 0 \) as \( n \to \infty \) for all \( x \in (0,1) \). Furthermore, given that \( f^{(n)}(0) = 0 \) for every \( n \in \mathbb{Z}_{\geq 0} \), it is not necessary to suppose that \( f \equiv 0 \) in \((-1,0)\). This can be shown analogously to \( f \equiv 0 \) in \((0,1)\).

**Problem 2020-3B** (proposed by Onno Berrevoets)

Consider the map \( f: \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0} \) \((a,b) \to (2 \min\{a,b\}, \max\{a,b\} - \min\{a,b\})\).

We call \((a,b) \in \mathbb{Z}_{\geq 0} \) equipotent if there exists \( n \in \mathbb{Z}_{\geq 0} \) such that \( f^n(a,b) = (x,x) \) for some \( x \in \mathbb{Z}_{\geq 0} \) (where \( f^n = f \circ \cdots \circ f \)). Show that \((a,b) \in \mathbb{Z}_{\geq 1}^2 \) is equipotent if and only if \( \frac{a+b}{\gcd(a,b)} \)

is a power of 2.

**Solution** We received solutions by Pieter de Groen and Ludo Pulles. This solution is based on the one by Pieter.

It is clear that \( f(ca,cb) = cf(a,b) \) for all non-negative integers \( a, b, k \), and it follows that \((ca,cb)\) is equipotent if and only if \((a,b)\) is. So it suffices to show that for \( a, b \geq 1 \) relatively prime, we have \((a,b)\) is equipotent if and only if \( a + b = 2^k \) for some \( k \geq 1 \).

\( \Rightarrow \): Suppose that \( a, b \in \mathbb{Z}_{\geq 1} \) are relatively prime and satisfy \( a + b = 2^k \). If \( k = 1 \), we have \((a,b) = (1,1) = f^0(1,1)\) is equipotent. If \( k > 1 \) and \( (c,d) := f(a,b) \), then \( c = 2 \min\{a,b\} \) is even, and hence so is \( d \) because \( c + d = a + b \) is even. Hence \((a,b)\) is equipotent with sum \( 2^k \) if and only if \( \left( \frac{c}{2}, \frac{d}{2} \right) \) is equipotent with sum \( 2^{k-1} \). Note that \( c \) and \( d \) are relatively prime because \( \gcd(2 \min\{a,b\}, \max\{a,b\} - \min\{a,b\}) \) can only take on the values \( \gcd(a,b) \) or \( 2 \gcd(a,b) \). We can conclude \((a,b)\) is equipotent by induction.

\( \Leftarrow \): Conversely, suppose that \( a, b \in \mathbb{Z}_{\geq 1} \) are relatively prime with \( a + b \) not a power of 2. Note that the sum of \((a,b)\) is invariant under \( f \), because if \( f(a,b) = (p,q) \), we have \( p + q = \max\{a,b\} + \min\{a,b\} = a + b \). If \( a + b \) is odd, then the same is true for \( f^n(a,b) \), so \((a,b)\) is not equipotent. Suppose \( a + b \) is even. Because \( a, b \) are relatively prime, both \( a \) and \( b \) are odd. Now similar to the above, if \( f(a,b) = (c,d) \), then both \( c \) and \( d \) are even, and \((a,b)\) is equipotent if and only if \( \left( \frac{c}{2}, \frac{d}{2} \right) \) is. Since \( \frac{c}{2} + \frac{d}{2} = \frac{a+b}{2} \) and \( \frac{c}{2}, \frac{d}{2} \) are relatively prime, repeating this procedure eventually results in a pair with odd element-sum, which is not equipotent. Hence \((a,b)\) was not equipotent either.

**Problem 2020-3/C** (folklore)

Uncle Donald cuts a 3 kg piece of cheese in an arbitrary, finite number of pieces of arbitrary weights. He distributes them uniformly randomly among his nephews Huey, Dewey and Louie. Prove or disprove: the probability that two of the nephews each get strictly more than 1 kg is at most two thirds.

**Solution** This problem remains open. This is a Star Problem for which the proposer does not know any solution. For the first correct solution sent in within one year there is a prize of €100.