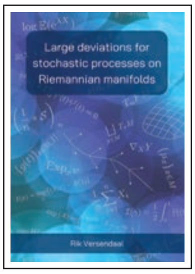
In de verdediging

| In defence



Large Deviations for Stochastic Processes on Riemannian Manifolds
Rik Versendaal

In September 2020 Rik Versendaal from the Delft University of Technology successfully defended his PhD thesis with the title *Large Deviations for Stochastic Processes on Riemannian Manifolds*. Rik carried out his research under the supervision of Prof. dr. Frank Redig, and Prof. dr. Jan van Neerven.

During his PhD Rik worked on large deviations for processes on Riemannian manifolds. In particular, he studied extensions of large deviations for random walks and Brownian motion on curved spaces. Furthermore, he also considered large deviations for random walks in Lie groups and large deviations for Brownian motion on evolving Riemannian manifolds, meaning that the Riemannian metric is time-dependent.

At the moment Rik is a postdoc at Utrecht University. Currently he is working on random (geometric) networks, both from a theoretical perspective but also on their practical applications.

Large deviation theory – the theory of the very rare

Large deviation theory is a mathematical theory, and a subfield of probability theory, that is concerned with quantifying the exponentially small probabilities of rare events, in particular deviations from the typical behaviour. Arguably the most well-known construction in probability theory is a random walk, i.e. a sum of independent random variables all having the same distribution. Since all these random variables have the same distribution they all have the same mean value and variance, denoted by μ and σ, respectively. The law of large numbers says that as the size of the sum grows, then the whole sum divided by the number of terms in the sum is almost equal to μ. A natural question that follows is whether we can say something about the deviations of this random walk from the typical behaviour as given by the law of large numbers. A first answer is given by the central limit theorem, which gives the 'normal' deviations. Moreover, the so-called large deviation principle provides the 'large' deviations. These three results are summarized in the Table 1, where $X_1, X_2, \dots$ denote independent and identically distributed random variables and $\mathcal{N}(0,1)$ denotes a random variable following the normal distribution with mean 0 and variance 1.

Where the law of large numbers only tells that the probability of large deviations goes to 0, the theory of large deviations is concerned with how fast this convergence is. Large deviations of random walks were first studied by Harald Cramér in the 1930s and a unified formalization of large deviation theory was developed later in the 1960s by the Indian mathematician and winner of the Abel Prize in 2010 Srinivasa Varadhan.

Pas gepromoveerden brengen hun werk onder de aandacht. Heeft u tips voor deze rubriek of bent u zelf pas gepromoveerd? Laat het weten aan onze redacteur.

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Law of Large numbers	Central Limit Theorem	Large Deviation Principle
$\frac{1}{n}\sum_{i=1}^{n} X_i \to \mu$	$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} \frac{X_i-\mu}{\sigma} \to \mathcal{N}(0,1)$	$\frac{1}{n}\log \mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} X_i \geq x\right) \to -\Lambda^*(x)$
$\sum_{i=1}^{n} X_i \approx n\mu$	$\mathbb{P}\left(\sum_{i=1}^{n} X_i \geq n\mu + \sqrt{n}\sigma x\right) \approx \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-\frac{y^2}{2}} dy$	$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n} X_i \geq x\right) \approx e^{-n\Lambda^*(x)}$
The random walk behaves almost like $n\mu$.	The deviations of order $\sqrt{n}$ of the random walk from the typical behavior $n\mu$ are given by the normal distribution.	The large deviations, i.e. of order n, of the random walk from the typical behavior $n\mu$ are given by the Legendre transform of the logarithmic moment generating function, denoted by $\Lambda^*(x)$.

Table 1

Walking on geodesics

The results discussed above concern random walks in Euclidean spaces, but what happens when you start walking in curved spaces? In his thesis Rik was motivated by this question and studied random walks on manifolds. This generalization is far from straightforward because even the basic definition of a random walk has to be adapted with care. Let's have a look into this. Indeed, if someone would simply copy the approach from the Euclidean case, a problem you immediately run into is that you cannot add points in a manifold together and rescale by a factor. This problem already occurs when one considers the sphere, which is the prototypical example of a manifold.

To set this right, the increments $\{X_n\}_{\{n\geq 1\}}$ of the random walk $\sum_{i=1}^{n} X_i$ may be thought of as vectors. The addition of such a vector then amounts to following the straight line in the direction of the vector for time 1 to assure that the entire vector is added. See the left picture in Figure 1 for a visualization of this interpretation. On a manifold, vectors providing directions are precisely the tangent vectors. Therefore, to make a 'step' of the random walk, a random tangent vector is considered. To make the next step the 'straight line' in that direction is followed. In Euclidean space, straight lines are lines of shortest distance between points, i.e., they are geodesics. On a manifold, following the 'straight line' means that the random walk has to follow the geodesic in that direction. The random walk is finally constructed by concatenating a number of random steps. This random walk after n steps is denoted by S_n and will be referred to as a geodesic random walk.

Harald Cramér

Srinivasa Varadhan

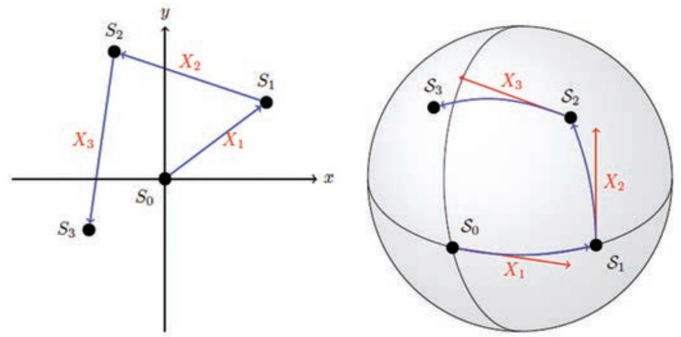

Figure 1 Visualization of the construction of a geodesic random walk. On the left, the interpretation of a random walk in Euclidean space as repeatedly following straight lines in the direction of vectors. On the right, this idea is extended to the sphere, where the walk follows geodesics in the direction of tangent vectors.

What remains is to define how the random walk can be rescaled by a factor $\frac{1}{n}$. This is done by rescaling the tangent vectors that yield the direction at which the walk moves. Equivalently, this corresponds to following the geodesics for time $\frac{1}{n}$ instead for time 1. The rescaled random walk is denoted by $(\frac{1}{n} * S_n)_n$.

The formal method to define a geodesic random walk relies on the Riemannian exponential map. Furthermore, because the space variable determines in which tangent space the increment should be, the random walk has to be defined recursively, this means that the direction of the n-th step is found given all the previous steps, namely S_{n-1}.

Two additional complications that arise in the Riemannian setting but not in the Euclidean, concern the proper definition of independent and identically distributed increments of the random walk. The crux is that the dependence of the direction of the n-th step and all the previous steps is purely geometric, in the sense that the previous steps of the random walk only determine the tangent space from which the new direction should be chosen. The second complication regards a proper definition of identically distributed increments. In general, the increments of a geodesic random walk do not live in the same tangent space. In order to overcome this, Rik relied on a technique called parallel transport, which was used to identify tangent spaces. Because the identification via parallel transport depends on the curve along which the vectors are transported the analysis is rather intricate.

In his thesis Rik managed to quantify the large deviations for scaled geodesic random walks on Riemannian manifolds. This result generalizes the previous known results for Euclidean spaces where the concepts of a random walk, an increment, a scaling and independence are rather straightforward to define. Besides random walks, Rik also studied another and more complicated stochastic process, the Brownian motion. A random walk as discussed above is a stochastic process in discrete time, you can imagine that the same questions can also be posed for stochastic processes in continuous time. In general, for every stochastic process where some typical behavior exists, think of a counterpart of the law of large numbers, you can ask the question of how the fluctuations around this typical behavior behave. The continuous-time stochastic process that Rik studied is the Brownian Motion.

In the Euclidean setting, Brownian motion $W(t)$ is usually defined as the unique continuous process with independent, stationary increments such that $W(t) - W(s)$ has a normal distribution with mean 0 and variance $t - s$. Since there is no clear way to define increments of a manifold-valued process, this approach is not suitable to define a Brownian motion on a manifold. It turns out that there are various methods, from geometric to probabilistic, to define a Brownian motion on a manifold if additional structure is considered. Rik managed to prove that for a Riemannian Brownian motion the probability it follows a certain path γ is exponentially small and the exponent is given by a rate function corresponding to the action of the path γ which depends on the Riemannian metric g.

Brownian motion on a Riemannian manifold – Pollen grains on a drop

In his thesis Rik also studied Brownian motion on a manifold that changes over time. More precisely, he considered manifolds with a Riemannian metric which changes over time. One can for example think of a sphere, whose radius varies in time. A motivation to proceed into this direction comes from molecular biology and the random movements of proteins in cell membranes. Cells usually deform over time, and this influences the stochastic process that describes the movement of the proteins. In the Euclidean case, the rate of a path has a physical interpretation, namely as an action of the path. Rik managed to prove the remarkable result that this interpretation is also true in Riemannian manifolds, even when equipped with a time-dependent metric.

The more personal aspect

Behind all dissertations there is always a person, with flesh and bones, who has endured the long path of a PhD trajectory and has produced the work at hand.

Were you also involved in some other activities and events during your PhD?

"I have done a research visit to the group of Anton Thalmaier in Luxembourg for two months. That is what helped me out in understanding how to deal with time-dependent geometry. Now I know the underlying idea where various 'magical' formula's come from. Furthermore, I was also a finalist for the KWG PhD Prize in 2019, where I got to present my PhD research in a short plenary talk at the NMC."

Are there some nice memories from the last four years you would like to share?

"It may not immediately sound like a nice memory, but it was a turning point in my PhD and a great triumph in the end. In my article on large deviations for geodesic random walks, I spotted a very tiny mistake in the proof when everything was as good as ready. Although the mistake was only one small equation, the consequences where huge, and I had to come up with quite some different arguments to make things work in the end. It was very frustrating, but the satisfaction was ever greater when I found the new approach that worked. It showed that even very small details can have a large impact."