Problemen

| Problem Section

This Problem Section is open to everyone; everybody is encouraged to send in solutions and propose problems. Group contributions are welcome. We will select the most elegant solutions for publication. For this, solutions should be received before **15 October 2020**. The solutions of the problems in this issue will appear in the next issue.

Problem A (proposed by Onno Berrevoets)

Let $f:(-1,1)\to\mathbb{R}$ be a function of class C^∞, i.e., all higher derivatives of f exist on $(-1,1)$. Let $c\geq 0$ be a real number. Suppose that for all $x\in(-1,1)$ and all $n\in\mathbb{Z}_{\geq 0}$ we have $f^{(n)}(x)\geq -c$. Also assume that for all $x\in(-1,0]$ we have $f(x)=0$. Prove that f is the zero function.

Problem B (proposed by Onno Berrevoets)

Consider the map $f:\mathbb{Z}^2_{\geq 0}\to\mathbb{Z}^2_{\geq 0},\ (a,b)\mapsto(2\min\{a,b\},\max\{a,b\}-\min\{a,b\})$.
We call $(a,b)\in\mathbb{Z}^2_{\geq 0}$ *equipotent* if there exists $n\in\mathbb{Z}_{\geq 0}$ such that $f^n(a,b)=(x,x)$ for some $x\in\mathbb{Z}_{\geq 0}$ (where $f^n=f\circ\cdots\circ f$). Show that $(a,b)\in\mathbb{Z}^2_{\geq 1}$ is equipotent if and only if $\frac{a+b}{\gcd(a,b)}$ is a power of 2.

Problem C* (folklore)

Uncle Donald cuts a 3 kg piece of cheese in an arbitrary, finite number of pieces of arbitrary weights. He distributes them uniformly randomly among his nephews Hewey, Dewey, and Louie. Prove or disprove: the probability that two of the nephews each get strictly more than 1 kg is at most two thirds.

Edition 2020-2 We received solutions from Rik Biel and Thijmen Krebs.

Problem 2020-2/A (proposed by Hendrik Lenstra)

Let R be a ring, and write $R[[X,X^{-1}]]$ for the set of formal expressions $\sum_{i\in\mathbb{Z}}a_iX^i$ with all $a_i\in R$.

a. Suppose that $R[[X,X^{-1}]]$ has a ring structure with the following three properties.
   i. The sum is given by $\left(\sum_{i\in n\mathbb{Z}}a_iX^i\right)+\left(\sum_{i\in\mathbb{Z}}b_iX^i\right)=\sum_{i\in\mathbb{Z}}(a_i+b_i)X^i$,
   ii. For two formal power series in X, the product is the regular product of power series, and likewise for two formal power series in X^{-1}.
   iii. For $1:=X^0$, we have $X\cdot X^{-1}=1$.

   Prove that R is the zero ring.
b. Prove that for every ring R, there exists a ring structure on $R[[X,X^{-1}]]$ satisfying properties I and II.

Solution We received a solution from Thijmen Krebs.
For part a, we calculate

$$\begin{aligned}\Big(\sum_{i>0}X^i\Big)\cdot\Big(\sum_{i<0}X^i\Big)&=\Big(\sum_{i\geq 0}X^i\Big)\cdot X\cdot X^{-1}\cdot\Big(\sum_{i\leq 0}X^i\Big)\\&=\Big(\sum_{i\geq 0}X^i\Big)\Big(\sum_{i\leq 0}X^i\Big)\\&=1+\Big(\sum_{i<0}X^i\Big)+\Big(\sum_{i>0}X^i\Big)+\Big(\sum_{i>0}X^i\Big)\cdot\Big(\sum_{i<0}X^i\Big)\\&=\sum_{i\in\mathbb{Z}}X^i+\Big(\sum_{i>0}X^i\Big)\cdot\Big(\sum_{i<0}X^i\Big).\end{aligned}$$

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It follows that the formal power series $\sum_{i\in\mathbb{Z}} X^i$ equals 0, and in particular, $1=0$. Hence R is the zero ring.

For part b, we define

$$\Big(\sum_{i\in\mathbb{Z}} a_i X^i\Big)\cdot\Big(\sum_{i\in\mathbb{Z}} b_i X^i\Big) = \Big(\sum_{i\geq 0} a_i X^i\Big)\Big(\sum_{i\geq 0} b_i X^i\Big) + \Big(\sum_{i\leq 0} a_i X^i\Big)\Big(\sum_{i\leq 0} b_i X^i\Big) - a_0 b_0.$$

This satisfies property (ii), has multiplicative identity 1, and is associative and distributive because multiplication in $R[[X]]$ and $R[[X^{-1}]]$ is associative and distributive.

Note that this ring is isomorphic to $R[[X,Y]]/(XY)$.

Problem 2020-2/B (proposed by Onno Berrevoets)

Let $n\geq 1$ be an integer, and let $p_1,\dots,p_{n-1}$ be pairwise distinct prime numbers. Suppose that $(v_1,\dots,v_n)^\top\in\mathbb{Z}^n$ is a non-trivial element of the kernel of

$$\begin{pmatrix} 1^{p_1} & 2^{p_1} & \dots & n^{p_1} \\ 1^{p_2} & 2^{p_2} & \dots & n^{p_2} \\ \vdots & \vdots & \ddots & \vdots \\ 1^{p_{n-1}} & 2^{p_{n-1}} & \dots & n^{p_{n-1}} \end{pmatrix}.$$

Prove that

$$\max_k |v_k| \geq \frac{2}{n^2+n}\prod_{i=1}^{n-1} p_i.$$

Solution We received no solutions. The proof provided is by Daan and Onno.

Suppose by contradiction that $\max_k|v_k|<\frac{2}{n^2+n}*\prod_i p_i$. For every i the i-th row vector equals $(1,2,\dots,n)$ modulo p_i, hence $\sum_k kv_k\equiv 0 \bmod p_i$ for every i. It follows that $\sum_k kv_k = 0 \bmod \prod_i p_i$, and thus $\sum_k kv_k=0$ by the assumed inequality on $\max_k|v_k|$. We conclude that $(1,2,\dots,n)$ is perpendicular to $(v_1,\dots,v_n)$. Now $(v_1,\dots,v_n)^\top$ is in the kernel of the matrix.

$$\begin{pmatrix} 1 & 2 & \dots & n \\ 1^{p_1} & 2^{p_1} & \dots & n^{p_1} \\ 1^{p_2} & 2^{p_2} & \dots & n^{p_2} \\ \vdots & \vdots & \ddots & \vdots \\ 1^{p_{n-1}} & 2^{p_{n-1}} & \dots & n^{p_{n-1}} \end{pmatrix}.$$

Hence, this matrix is singular, and the rows are thus linearly dependent over $\mathbb{Q}$. Let $S=\{1,p_1,p_2,\dots,p_{n-1}\}$ and let $(\lambda_s)_{s\in S}\in\mathbb{Q}^n$ be such that

$$\sum_{s\in S}\lambda_s(1^s,2^s,\dots,n^s)=(0,0,\dots,0).$$

Then the non-zero polynomial $\sum_s \lambda_s X^s$ has roots $1,2,\dots,n$ and at most n non-zero coefficients.

We claim that such a polynomial does not exist. More precisely, we claim that any non-zero polynomial in $\mathbb{R}[X]$ with at least n distinct positive roots has at least $n+1$ non-zero coefficients. We do this by induction.

The case $n=0$ is trivial. For $N>0$, assume by induction that any non-zero polynomial with at least $N-1$ distinct positive roots has at least N non-zero coefficients. Suppose that f is a non-zero polynomial with at least N positive roots. Without loss of generality, assume the constant term of f is non-zero (dividing by X does not change the number of terms or the number of positive roots of f), and moreover, f is non-constant. Suppose f has positive roots $a_1<a_2<\dots<a_N$. Then the derivative of f has a positive root between a_i and a_{i+1} for all $i\in\{1,2,\dots,N-1\}$, so it has at least $N-1$ distinct positive roots. By the induction hypothesis, it has at least N non-zero coefficients. Since f had a non-zero constant term by assumption, it follows that f must have had at least $N+1$ non-zero coefficients to begin with.

This proves the claim, and shows in particular that there is no non-zero polynomial with at most n non-zero terms with roots $1,2,\dots,n$. This contradiction shows that we cannot have $\max_k|v_k|<\frac{2}{n^2+n}*\prod_i p_i$, as was to be shown.

Problem 2020-2/C (proposed by Onno Berrevoets)

Let $n, m, k \geq 2$ be positive integers. n students will attend a multiple-choice exam containing mk questions and each questions has k possible answers. A student passes the exam precisely when he/she answers at least $m+1$ questions correctly.

a. Suppose that $n = 2k$. Show that the students can coordinate their answers such that it is guaranteed that at least one student passes the exam.
b. Suppose that $n = 2k-1$. Does there exist a k for which the students can coordinate their answers such that it is guaranteed that at least one student passes the exam?

Solution We received solutions from Rik Biel and Thijmen Krebs. This solution is based on the one by Thijmen.

We describe a strategy for the students by giving mk ordered partitions of $\{1, \ldots, n\}$ into k parts, indicating that student s picks answer a on question q if and only if s is in the a-th part of the q-th partition.

For part a, we take

$$\begin{aligned} &\{1,2\},\{3,4\},\ldots,\{n-1,n\} && \text{(first } mk-1 \text{ questions)},\\ &\{2,3\},\{4,5\},\ldots,\{n,1\} && \text{(final question)}. \end{aligned}$$

Note that student $2i$ is in the i-th part of each partition, and that student $2i-1$ is in the i-th part of each partition except the last, in which case she is in the $i-1$-th part (taken cyclically). In other words, student $2i$ answers i on each question, while student $2i-1$ answers i on all but the last question, on which she answers $i-1$.

The only way in which none of the students passes on the first $mk-1$ questions is if all but one of the student pairs $\{2i-1, 2i\}$ get exactly m correct answers and the final pair gets exactly $m-1$ correct answers on these questions. Assuming this is the case, then if the correct answer to the final question is i, we find that both students $2i$ and $2i+1$ get another correct answer, and at least one of these two already had m correct answers on the first $mk-1$ questions. Thus, at least one student passes.

For part b, there exists such k if $k > m+2$. Take

$$\begin{aligned} &\{1,2\},\{3,4\},\ldots,\{n-2,n-1\},\{n\} && m(k-1)-1 \text{ times},\\ &\{2,3\},\{4,5\},\ldots,\{n-1,1\},\{n\} && m+1 \text{ times}. \end{aligned}$$

Without loss of generality, we assume student n is correct on the first i questions and the last $m-i$ questions with $0 \leq i \leq m$; if she is correct more often, we are done, and otherwise, the situation can only improve for the remaining students. Then there are $m(k-1)-(i+1)$ questions of the first kind and $1 \leq i+1 < k-1$ questions of the second kind for the other students to score points on. On each of these questions, precisely two students score points, for a total of $2m(k-1)$ points distributed over $2(k-1)$ students. This means that no student passes if and only if each student scores exactly m points.

The only way to avoid passing a student on the remaining $m(k-1)-(i+1)$ questions of the first kind is for no fewer than $k-1-(i+1)$ of the $k-1$ pairs $\{1,2\}, \ldots, \{n-2, n-1\}$ to have exactly m correct answers; otherwise, the total number of questions of this kind can at most be $(k-1-(i+2))\,m + (i+2)\,(m-1) = m(k-1)-(i+2)$. The remaining $i+1$ pairs have in total $2((i+1)\,m-(i+1))$ correct answers. In particular, since $1 \leq i+1 < k-1$, there is at least one pair of students with exactly m points and at least one pair with strictly less than m points scored on the questions of the first kind. Working cyclically, we can therefore assume that the pair $\{2j-1, 2j\}$ has exactly m correct answers and $\{2j+1, 2j+2\}$ has strictly less than m correct answers in questions of the first kind. We find that student $2j$ cannot have any correct answer in the final questions, and since she answers in the same way as student $2j+1$, neither can student $2j+1$. But then student $2j+1$ will in total score strictly less than m points, meaning that at least one other student will score strictly more than $m+1$ points. In other words, this strategy guarantees that at least one student passes.