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Education

A glimpse into the Pre-University College

In April 2019 the two high school students Aryan Rezaei Ghavamabadi and Tobias Lewerissa received the best research prize of the Pre-University Program. In this article their supervisors Marta Maggioni and Matteo Sfragara give a general overview on the Pre-University Program in the Netherlands and emphasize the collaboration between Leiden University and various high-schools. The last part of the article presents the research that led to the prize and it is written together with the two students.

The Pre-University Program started almost fifteen years ago as an initiative of Leiden University and Utrecht University, aiming to offer the opportunity to high school students to follow an academic program focused on science and technology. The program is intended for secondary education students from vwo 5 and 6 and nowadays more universities participate. Although the content and the structure of the curriculum vary between universities, they all stimulate participants to engage into a research project.

The main leading figures of the Pre-University Program are Ayla Murad (project leader mbo-hbo network in Utrecht), Annebeth Simonsz (education coordinator and study advisor at Leiden University), Sanne Tromp (Junior College Utrecht director) and Ton van der Valk (former curriculum coordinator Junior College Utrecht).

Leiden University is part of the Pre-University Program, and it has been offering the so-called Pre-University College (PRE-College) since 2004, see [1]. The aim of this two-year educational program is mainly to help students who are interested in science to get acquainted with scientific reasoning and research. Every year ninety-five students are chosen from regional schools through a selection process where their grades averages, motivation and test results are taken into account.

The program consists of five different blocks (three in the first year and two in the second one), with lectures, tutorials, excursions and assignments. The first block consists of lectures on scientific knowledge, aiming to gain insight into the method(s) with which research is conducted and knowledge is built up in various scientific disciplines. In the second block students follow lectures from a wide range of areas and get acquainted with scientific research. During the third block students focus on the relationship between science and society. In this period the students also state their preference with regard to a research partner and a faculty field in which they would like to conduct a research project. In the fourth block students carry out their research in pairs under the guidance of university lecturers and they follow supporting meetings with all PRE-College members. The last block concentrates on the value of research outside academia.

The students

In 2018, Aryan Rezaei Ghavamabadi (Alkwin College Uithoorn) and Tobias Lewerissa (Stanislas College Delft) were two out of the four students to have chosen mathematics as the subject for their final research. They have worked for six months, with regular meetings at the Mathematical Institute of Leiden, under the supervision of Marta Maggioni and Matteo Sfragara, both PhD candidates in Mathematics at Leiden University.

“Something that fascinates me about mathematics is that it is the foundation of so many sciences, such as physics. Mathematics has taught me a lot about the universe, from quantum physics to cosmology and everything in between. With mathematics we can answer questions that might come up in everyday life”, says Aryan.

“High school mathematics can be quite monotonous and tedious. Exercises usually require knowledge of a concept that has just been introduced. On the other hand, when you do research, there is no theory of any kind to fall back. You need to be creative and come up with a strategy on your own. The most groundbreaking ideas always seem to come when they are least expected: under the shower, at the train station or during dinner. These breakthroughs,

The students receiving the Jan Kijne-onderzoeksprijs. From left to right: Marta, Aryan, Tobias and Matteo.

these intense moments of satisfaction, make it all worthwhile”, admits Tobias.

Aryan and Tobias chose probability theory and Markov chains as their research topic, and wrote an original research paper on ‘The knight errant problem’, part of which is presented in the next sections. Every year, at the end of the PRE-College program, a ceremony is held in Leiden, where all the students and their families are invited. It begins with some introductory speeches by the PRE-College organizers, giving an overall view of the program, and it ends with all the students receiving their PRE-College diploma. In between, the best research paper is awarded by a jury with the so-called *Jan Kijne-onderzoeksprijs*, see [2]. In April 2019, Aryan and Tobias won this prize (see the picture above), together with another pair of students (Naïm Achahboun and Florian Velten with a project in astronomy).

The knight errant and Markov chains

A knight walks randomly on a canonical empty chessboard. How many steps does it need to return to its original position?

The ramble of the knight can be modeled by a random walk. In a nutshell, a random walk describes a path obtained from successive random steps on some space. If each step only depends on the current position, i.e. it does not depend on the entire walk, the random walk is a Markov chain. The problem asks for the average amount of time needed to the knight to return to its initial position in the canonical chessboard. Recall that the move of a knight is given by one step in one direction followed by two steps in a direction that is perpendicular to the first, see Figure 1. When several steps are made, the random walk of the knight can be studied as a Markov chain $(X_n)_{n\in\mathbb{Z}_{\geq 0}}$, where X_n denotes the position of the knight after n steps on a graph of 64 nodes, which is the state space. Each of these nodes represents a square of the chessboard and there is an edge between two nodes if the corresponding squares are neighbors. See [3] for a general overview of the problem.

Below we show how the problem can be solved in two or even higher dimensions. We start by recalling some basic definitions and results about Markov chains.

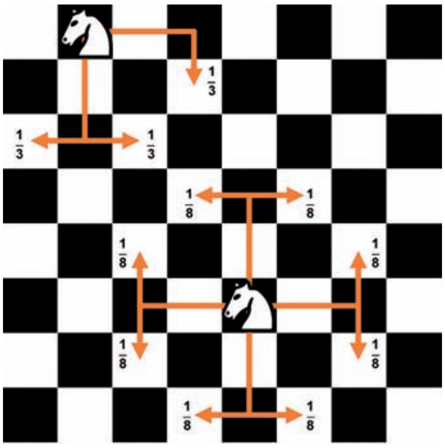

Figure 1 Some possible moves of the knight, depending on its position.

Definition 1 (Markov chain).

– A Markov chain on a set S is a stochastic process satisfying the Markov property, i.e., such that

$$\begin{aligned}&\mathbb{P}(X_n=i_n\,|\,X_{n-1}=i_{n-1},\ldots,X_0=i_0)\\&\quad=\mathbb{P}(X_n=i_n\,|\,X_{n-1}=i_{n-1}),\end{aligned}$$

for any $i_n,\ldots,i_0\in S$ and $n\in\mathbb{Z}_{>0}$.

– A stationary distribution $\boldsymbol{\pi}$ of a Markov chain is a probability distribution that remains unchanged as time progresses, i.e., such that

$$\boldsymbol{\pi}=\boldsymbol{\pi}P,$$

where P denotes the transition matrix of the chain.

– The return time of state i given $X_0=i$ is

$$T_i=\min\{n\geq 1\colon X_n=i\,|\,X_0=i\}.$$

The stationary distribution gives us information about the stability of the Markov chain and it describes its limiting behavior. Hence, the average return time for the knight is easily obtained by computing the stationary distribution of its Markov chain (it is known that the stationary distribution is unique under certain conditions which are satisfied in our case), and we get that for each state i,

$$E(T_i)=\frac{1}{\boldsymbol{\pi}_i}.$$

Indeed, let the degree of a node be the number of possible moves of the knight from the square corresponding to that node. Then, the sum of the degrees of all the nodes divided by the degree of a node i gives the average return times of i. In a canonical chessboard, it is pretty easy to compute both these ingredients: the sum of all the degrees, and the degree of a node. Look at Figure 1: the knight in b8 has three possible moves, hence the degree of the node b8 is 3; similarly eight choices are possible for the knight in e3, whence the degree of e3 is 8; and so on for each of the 64 nodes (and, in a similar fashion, both quantities can also be analysed for other pieces: the rook, the bishop, the king or the queen).

However, if the canonical chessboard is replaced with a less common one, given by a cube in $\mathbb{R}^d$, the situation becomes quite complex. Not only you can not picture anymore the moves of the knight in $\mathbb{R}^d$, for $d>3$, but also the degrees of the node increase, since the higher the dimension the more available moves the knight has.

The knight errant in d-dimension
Let $B_{d,r}$ be a cube of side r in $\mathbb{R}^d$.

Theorem 1 (Average return time). *The average return time of a knight in $B_{d,r}$ starting from a node i is given by*

$$\mathbb{E}(T_i) = \frac{4d(d-1)(r-1)(r-2)\,r^{d-2}}{4d^2-(4p+2q+4)d+(p+q)(p+2)+p}.$$

where p is the number of borders with distance 1 to i and q is the number of borders with distance 2 to i.

Proof. The proof consists of two results which combined together give the average return time.
1. An allowed move for a knight consists of one step in a direction (which we call primary) followed by two steps in another direction perpendicular to the first (which we call secondary). The product between the number of possible choices for the primary direction and the secondary direction determines the number of allowed moves. Note that while the secondary direction always needs to be different from the primary, both directions present further constraints due to the distance to the border. More precisely, at every step the knight has to be in the state space, i.e. he should not leave $B_{d,r}$. These constraints depend on the initial position, and are described by the parameters p and q. These considerations lead to:

$$\begin{aligned}\deg(i) &= 4d^2-(4p+2q+4)\,d\\ &\quad +(p+q)\,(p+2)+p.\end{aligned}$$

2. Consider, e.g., $B_{3,4}$ and suppose a knight starts in one of the corners of the cube, with coordinates $(4,1,4)$. It is as if the knight is present in three distinct 2-dimensional boards of side 4 at the same time, all of which are facing a different direction. The knight has two possible moves in every board and these transitions that lie within the 2-dimensional boards are the only possible transitions in $B_{3,4}$ (see Figure 3). The knight can never leave all three boards in a single L-shaped step, since it only travels along two axes when it makes its jump. The moves of a knight in $B_{3,4}$ can be seen as the sum of its moves in three distinct

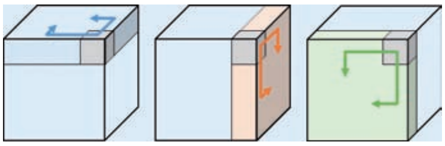

Figure 3 Moves of a knight in $B_{3,4}$.

2-dimensional boards, all of which are facing a different direction. Hence, the degree of a particular node in $B_{3,4}$ can be seen as the sum of the degrees of the same node in all 2-dimensional boards. The total degree $N_{3,4}$ of $B_{3,4}$ is obtained by summing over the total degrees of twelve 2-dimensional boards.

The total degree $N_{d,r}$ can thus be calculated by multiplying the total degree $N_{2,r}$ by the number of directions a 2-dimensional board can face in, say k, and by the number of times each board is counted, say l. A 2-dimensional board has two axes and by simply drawing two out of d axes, one of the directions of a 2-dimensional board can be obtained. Hence, k is equal to the number of ways of drawing an unordered pair out of a set of d axes, which gives

$$k = \binom{d}{2} = \frac{1}{2}d(d-1).$$

To find l, one can divide the number of nodes in $B_{d,r}$ by the number of nodes in $B_{2,r}$, which gives

$$l = \frac{r^d}{r^2} = r^{d-2}.$$

We then obtain

$$N_{d,r} = \tfrac{1}{2}d(d-1)\,r^{d-2}N_{2,r}.$$

To find $N_{2,r}$ we distinguish nodes based on their distance to both borders and we compute the degree and count the number of each type of node in $B_{2,r}$. By multiplying the degrees by the respective counts and by summing over these values, we get

$$N_{2,r} = 8(r-1)\,(r-2).$$

The sum of the degrees of all nodes in $B_{d,r}$ is then given by

$$N_{d,r} = 4d(d-1)\,(r-1)\,(r-2)\,r^{d-2}. \quad \square$$

In Table 1 we present the results of a computer simulation obtained by a program in Python. The program evaluates the return times by having a knight perform a trial run. A knight starts on one of the corners of $B_{d,r}$ and the program keeps count of its jumps until it returns to the initial position. The program runs several times and the average of all the outcomes evaluates the average return time. For different values of d and r the outcomes have also been compared with the theoretical result of Theorem 1.

Conclusions
The scientific journey and the research project the authors went through has been very stimulating and satisfying. Moreover, Aryan and Tobias enjoyed so much the PRE-College and their first real jump into the mathematics ocean, that they will continue studying mathematics at Leiden University.

d	r	1	2	3	4	5	Av. value	Th. value
2	5	51.74	38.96	43.78	52.30	54.56	48	48
2	6	72.14	80.26	81.86	83.12	78.70	79	80
2	7	119.92	102.06	112.80	127.20	103.38	113	120
2	8	171.34	174.68	170.88	161.58	146.12	165	168
3	5	240.10	261.10	249.76	235.44	206.18	238	240
3	6	525.72	476.12	423.96	440.60	508.28	474	480
3	7	865.00	795.32	722.44	1051.80	818.36	850	840
3	8	1172.42	1319.64	1419.88	1430.32	1725.06	1413	1344

Table 1 Outcomes of the program performing 5 sessions of 100 trial runs, with average compared with theoretical result.

References
1 Pre-University College website: https://www.universiteitleiden.nl/pre-college.
2 Jan Kijne-onderzoeksprijs article: https://www.universiteitleiden.nl/pre-college/nieuws/twee-duos-winnen-jan-kijne-onderzoeksprijs.
3 Can a Chess Piece Explain Markov Chains?, Youtube, https://youtu.be/63HHmjlh794.