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In Memoriam Anne Sjerp Troelstra (1939–2019)

Creating order in a vast and diverse area

On 7 March 2019 Anne Sjerp Troelstra passed away at the age of 79. Troelstra was emeritus professor at the Institute for Logic, Language and Computation of the University of Amsterdam. He dedicated much of his career to intuitionistic mathematics and wrote several influential books in this area. His former colleagues Johan van Benthem and Dick de Jongh look back on his life and work.

Anne Troelstra was born on 10 August 1939 in Maartensdijk. After obtaining his gymnasium beta diploma at the Lorentz Lyceum in Eindhoven, in 1957, he enrolled as a student of mathematics at the University of Amsterdam — where eventually his interests converged on intuitionism with Arend Heyting as his teacher. Students he was close with include Olga Bakker and E.W. Beth's students Dick de Jongh and Hans Kamp. With Dick de Jongh, he wrote a pioneering paper on intuitionistic propositional logic, published only in 1966, that contained the first definition of the central notion of a p -morphism between relational frames, widely used today in intuitionistic and modal logics, as well as the simplest form of the duality between Heyting algebras and relational frames. After obtaining his master's degree in 1964, Anne at once became an assistant professor, according to a custom of the time. It took him just two years from there on to finish his dissertation, supervised by Heyting. Besides intuitionism, a main interest of Heyting was geometry and perhaps not accidentally, Anne's PhD thesis was a study of intuitionistic topology. This subject made him aware of the role of continuity in intuition-

istic mathematics, a concept that was to play an important part in his research in the years to come, in many different forms.

Stanford

Anne then obtained a scholarship to Stanford to visit Georg Kreisel, a leading scholar in intuitionism and proof theory, and spent

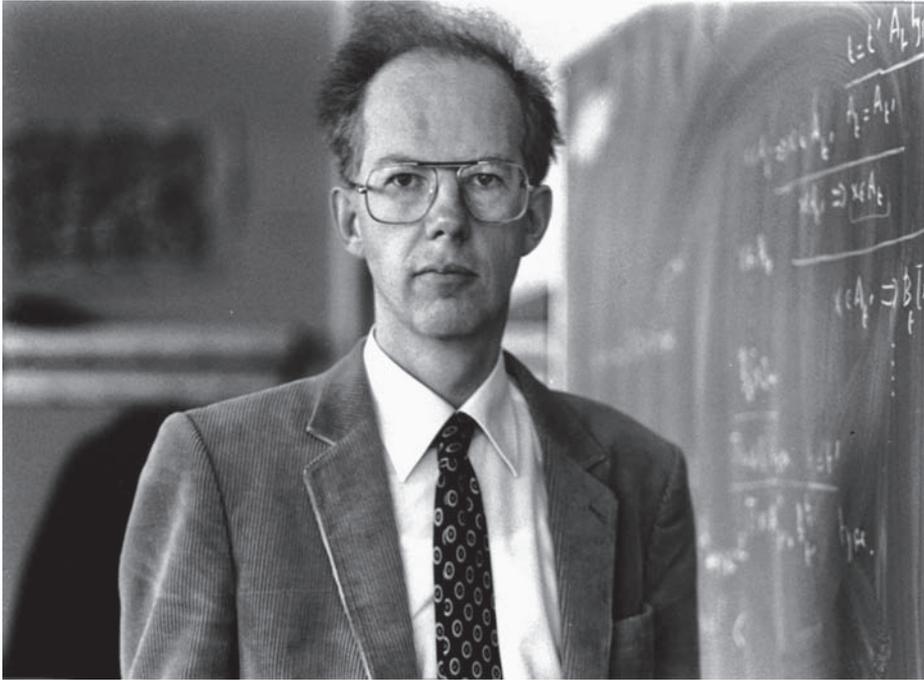


Anne Troelstra

the academic year 1966–1967 there, together with Olga whom he had married the year before. In this period, Anne sharpened and modified Kreisel's ideas on choice sequences, the basic notion underlying the real number continuum in an intuitionistic perspective. Working together they created formalizations of analysis resulting in a large article where the typically intuitionistic concept of a lawless sequence of numbers that successfully evades description by any fixed law, reached its final form. Though the latter notion is very characteristic for intuitionistic mathematics as based on free creation of objects, lawless sequences are one of the rare concepts in intuitionism not already introduced by Brouwer himself.

Constructivism

In August 1968, Anne played a central role in the famous Buffalo Conference on Intuitionism and Proof Theory, a meeting of all important logicians of the day with an interest in constructivism. He gave a series of ten lectures there, which turned into his first book, published in 1969, that contained the core of his seminal ideas on intuitionistic formal systems and their metamathematical investigation. Back home in 1968, he became a lector (associate professor) in 1968, and a full professor in 1970. Further early recognition was to follow. In 1976, he became a member of the Dutch Royal Academy of Sciences. As a full pro-



Anne Troelstra in 1985

fessor, Anne was the successor of Arend Heyting, himself a successor to L.E.J. Brouwer. Each was in his time the principal constructivist in the world.

Realizability

The metamathematics of intuitionistic systems was a chaotic jumble of results when Anne entered it. Here he showed his greatest strength: creating order in a vast and diverse area. In 1973, the rigorous order was there in his book *Metamathematical Investigation of Intuitionistic Arithmetic and Analysis*. Especially striking are the clarification of the properties of different models for constructive mathematics, several types of realizability, and various functional interpretations. The last chapters on special topics were written by Jeff Zucker, Craig Smorynski and Bill Howard, but the lion's share had been written by Anne, editor and architect of the whole. The book, known in the community as 'Springer 344', still functions as a landmark for researchers in the subject.

The metamathematical style of investigation in this work represents a significant broadening of earlier research. In tandem with developing branches of intuitionistic mathematics such as constructive arithmetic, analysis, topology or algebra, one now also studies the resulting formal theories as mathematical objects, using the tools of logic such as proof theory, recursion theory,

and model theory. Perhaps the most prominent among the many results obtained by Anne in this style concern 'realizability', a mathematical analysis of constructive truth introduced by Stephen Kleene in the 1940s. Its purpose was to explain what is 'constructive' about constructive mathematics in terms of recursive functions on natural numbers, then and now a standard paradigm for effective computability. Long viewed as an outside look at intuitionism from the standpoint of classical logic, Anne took realizability to the heartland of intuitionism. Anne's studies of the topic had already resulted in a long article in the proceedings of the Second Scandinavian Logic Symposium of 1971, and they were now expanded in Springer 344.

Troelstra's celebrated theorem characterizing realizability states that the formulas that are provably realizable in Heyting Arithmetic (intuitionistic natural number theory, arguably the most basic formal system of intuitionistic mathematics) are exactly the ones provable from Heyting's Arithmetic plus the so-called Extended Church Thesis. This result displays some typical features of the metamathematical style of investigation. First, the equivalence is formulated in a proof-theoretic setting. Secondly, informal claims such as Church's Thesis saying that every intuitively computable function can be computed by a recursive function, or equivalently a Turing machine, are for-

mulated within number theory. Church's Thesis says in this format that, for any true intuitionistic quantifier combination $\forall x \exists y B(x, y)$ (the typical sort of statement where constructive mathematics demands a procedure to produce the y 's from the x 's), there exists a recursive function f such that $\forall x B(x, f(x))$. From an informal thesis, Church's thesis has now become a precise axiom that can be true or false in intuitionistic mathematics. In the Extended Church Thesis the universal quantifier is relativized to antecedents A that have to be special 'almost negative formulas', untouched by realizability: If $\forall x (A(x) \rightarrow \exists y B(x, y))$, then $\forall x (A(x) \rightarrow B(x, f(x)))$ for some recursive function f . This is a typical form of careful metamathematical attention to syntactic fine-structure of mathematical assertions that has become standard in the subsequent literature.

Troelstra's result was the first of its kind, and it is widely seen as making the force and meaning of realizability clear within the context of intuitionism. The influence of this work shows in a broad literature analyzing other important notions of realizability with similarly structured characterizations. This study of realizability later developed broader mathematical connections with category theory and topos theory. An interest in this topic remained with Anne for life. In 1998, a chapter on realizability by his hand came out in the *Handbook of Proof Theory*. Characteristically, Anne's text had been finished a few years before, faithfully meeting his deadline, but delays by other authors kept him updating, somewhat grumblingly, with all new results in the area. What he published, on this and in fact on any topic, had to be the complete state of the art.

Choice sequences

A next landmark was Anne's book *Choice Sequences* from 1977, exploring the heartland of intuitionistic analysis in mathematical depth and with conceptual finesse. A principal result from this book is Troelstra's Elimination Theorem, which says, essentially, that any statement quantifying over lawless sequences in formalized intuitionistic theories is provably equivalent to a statement that is only about lawlike sequences and objects. This is reminiscent of well-known elimination results in mathematics concerning, say, the reduction from numbers to sets, though the proof

in Troelstra's case is more complex. Later, he went on to prove similar theorems for other types of choice sequences. The Elimination Theorem clarifies the position of lawless and choice sequences in important formal systems, but it does not make the concept of a lawless sequence lose its value: lawless and other choice sequences remain crucial to guarantee a really intuitionistic analysis. Troelstra's result rather means that these concepts can be based on a theory that does not contradict standard mathematics. How the concept of choice sequence remains crucial is shown very clearly by Anne in an extensive article in 1983 in the *Journal of Philosophical Logic*. He shows how lawless sequences and other choice sequences arise by informal, yet rigorous concept analysis (often called 'informal rigor'), the main source of knowledge in intuitionistic mathematics in general and about choice sequences in particular.

History of intuitionism and proof theory

Other topics pursued in depth by Anne from the 1970s onward are the history of intuitionism and its philosophical basis. Over time, all this work, consolidated in his monographs and articles, fed into the magisterial *Constructivism in Mathematics*, in two volumes co-authored with Dirk van Dalen, published in 1988, which remains the standard text on constructivism right until today.

Moving beyond intuitionism proper over the years, Anne broadened his scope to proof theory in general and wrote two more books that again created new order in developing fields. In 1992, his textbook *Lectures on Linear Logic* proposed im-

proved formats for the then still only partially understood new paradigm of linear logic merging proof theory and computation. Anne also contributed the majority of the chapters in a book with Helmut Schwichtenberg called *Basic Proof Theory* (1996) that is still a standard resource.

Students

An important part of Anne's life were his PhD students, of whom he supervised seventeen, and with many of whom he maintained a close relationship. His first PhD student Daniel Leivant finished a thesis in 1975 on the metamathematics of intuitionistic arithmetic, and later made his career in computer science. Initially a scarce commodity, in the 1980s, the number of PhD students increased, and Anne's students wrote on a broad variety of topics, such as intuitionistic metamathematics, combinatory algebra, category theory, Martin-Löf type theory, bounded arithmetic, linear logic, and provability logic. Many of these topics reflected the introduction by Anne, often in a close collaboration with Dirk van Dalen in Utrecht, of new topics on the Dutch scene. These students then carried the torch further by themselves. For instance, Ieke Moerdijk became an international leader at the interface of topos theory, logic and category theory generally, while Jaap van Oosten became a worldwide authority on realizability. In the Netherlands alone, four of Anne's students have become full professors, in mathematics, computer science, AI and philosophy. But Anne was a dedicated teacher at all academic levels, whose precision, clarity and scholarship influenced generations of students in Amsterdam.

Natural history

Anne retired in 2000, but not to rest. All his life, he had a deep interest in natural history and a wide knowledge of the plants of the Netherlands and abroad. His annual linocuts of plants discovered on his travels in Europe were famous. To those on a walk with Anne, listening to him turned what looked like an ordinary city street to the untrained eye into a rich landscape of flora, history, and natural wonders. This very year 2019, an article by him will appear on new species of blackberries, his special interest. Anne also made a further name for himself as an author on natural history travel narratives, chronicling the exotic characters and adventures of the past denied to the average academic of today. *Tijgers op de Ararat* from 2003 is a typical sample of this mix of information and human interest. Characteristically, he managed to create order even in this new field. His major *Bibliography of Natural History Travel Narratives* was published in 2016.

A very special person

Anne will be missed in the first place because he will no longer be there to tell us what he thinks about a question that you may have about constructivism. You always knew that you would get a completely honest answer from somebody who knew all the issues and had already thought much further than you. But it is just as much the personal qualities that will be missed. Anne was a very special, and to some, occasionally intimidating person: penetrating, honest, critical, ironic, sharp at times, but always open to arguments and unfailingly supportive of his students and colleagues. He will be deeply missed by all. ☘



Anne Troelstra in 1988 with Boris Kushner from Moscow (later Pittsburgh)



Anne Troelstra in 1998 with Justus Diller from Münster