Problemen

| Problem Section

This Problem Section is open to everyone; everybody is encouraged to send in solutions and propose problems. Group contributions are welcome. *We no longer reward a book token for solutions, but we will still select the most elegant solutions for publication.* The solutions of the problems in this issue will appear in the next issue.

Problem A (proposed by Arthur Bik)

Let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \in \mathrm{SO}(3)$$

be a matrix not equal to the identity matrix. Prove: if the vector

$$\begin{pmatrix} (a_{23}+a_{32})^{-1} \\ (a_{13}+a_{31})^{-1} \\ (a_{12}+a_{21})^{-1} \end{pmatrix}$$

exists, then A is a rotation using this vector as axis.

Problem B (proposed by Onno Berrevoets)

1. Let $k \in \mathbb{Z}_{>0}$ and let $\mathcal{X} \subset 2^{\mathbb{Z}}$ be a subset such that for all distinct $A,B \in \mathcal{X}$ we have $\#(A \cap B) \le k$. Prove that $\mathcal{X}$ is countable.
2. Does there exist an uncountable set $\mathcal{X} \subset 2^{\mathbb{Z}}$ such that for all distinct $A,B \in \mathcal{X}$ we have $\#(A \cap B) < \infty$?

Problem C (proposed by Onno Berrevoets)

Let $A : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function such that for every $x,y,z \in \mathbb{R}$ we have
1. $A(x,y) = A(y,x)$,
2. $x \le y \Rightarrow A(x,y) \in [x,y]$,
3. $A(A(x,y),z) = A(x,A(y,z))$,
4. A is not the $\max$ and not the $\min$ function.

Prove that there exists an $a \in \mathbb{R}$ such that for all $x \in \mathbb{R}$ we have $A(x,a) = a$.

Edition 2019-1 We received solutions from Paul Hutschemakers, Hendrik Reuvers and Hans Samuels Brusse.

Problem 2019-1/A (folklore)

Three gamblers each select a non-negative probability distribution with mean 1. Say these distributions are F, G, H. Then x is sampled from F, y is sampled from G, and z is sampled from H. Biggest number wins. What distributions should the gamblers choose?

Solution Suppose two of the gamblers choose the same distribution function: $\Phi(t) = \sqrt{t/3}$ on the interval $[0,3]$. What should the other gambler do? If she flips a coin and says 3 for H and 0 for T, then she needs a coin that has probability $\frac{1}{3}$ for H in order to comply with the rules of the game. She wins one third of the time. She could also try a fair coin and say 2 for H and 0 for T. What is the probability that she wins with this strategy? She needs to beat the maximum of two numbers that are sampled from the $\sqrt{t/3}$-distribution. The distribution of the maximum is $M(t) = t/3$. Therefore, the probability that 2 is the winning number is $\frac{2}{3}$. The probability that H comes up is $\frac{1}{2}$. Again, she wins one third of the time.

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Of course, the other gambler may try other coins and other numbers. We leave it to the reader to verify that each of them have a probability of $\frac{1}{3}$ of winning. The other gambler should not use a number bigger than three. If she is overly concerned and says 4 just to be on the safe side, she will not win one third of the time.

Any probability distribution on $[0,3]$ with mean one is a mixture of coins with mean one. Therefore, the other gambler may just as well sample from $\Phi(t)$ to win one third of the time. If all three gamblers sample from this distribution, each of them wins a third of the time and has no reason to deviate. We solved the game. As always, it is a bit of a mystery how we found this solution. There is no good algorithm to find a Nash equilibrium.

This game is taken from a recent paper by Steve Alpern and John Howard, 'Winner-take-all-games', *Operations Research* 65, 2017. They solve the n-player version and show that the solution is unique. Apparently, it remains an open problem to solve the game if different players have different means. Suppose we have a new Da Vinci coming up at Christie's and three different Saudi royals with three different means want to buy it. In a one-shot auction, how should they bid?

Problem 2019-1/B (proposed by Hendrik Lenstra)

For given $m \in \mathbb{Z}_{\geq 3}$, consider the regular m-gon inscribed in the unit circle. We denote the surface of this m-gon by A_m. Suppose m is odd. Prove that $2A_m$ and A_{2m} have the same minimal polynomial.

Solution We find $A_m = \frac{m}{2}\sin(\frac{2\pi}{m})$ by basic geometry, so $2A_m = m \cdot \sin(\frac{2\pi}{m})$ and $A_{2m} = m \cdot \sin(\frac{2\pi}{2m})$. Let ζ be a primitive $2m$-th root of unity. If we embed in $\mathbb{C}$ by taking $\zeta = \cos(\frac{2\pi}{2m}) + i\sin(\frac{2\pi}{2m})$ in $\mathbb{C}$, we find $A_{2m} = \frac{m}{2i}(\zeta + \zeta^{-1})$ and $2A_m = \frac{m}{2i}(\zeta^2 + \zeta^{-2})$. Since m is odd, ζi is a primitive $4m$-th root of unity, and so is $\zeta^2 i$. We consider the field $\mathbb{Q}(\zeta, i) = \mathbb{Q}(\zeta i)$. Observe that the field automorphism defined by sending ζi to $\zeta^2 i$ sends ζ to $-\zeta^2$ and i to $-i$ (this can be verified using $i = \pm(\zeta i)^m$ and $\zeta = \pm(\zeta i)^{m+1}$). Therefore this automorphism sends A_{2m} to $2A_m$ (implicitly using the earlier embedding into $\mathbb{C}$). Since automorphisms preserve minimal polynomials, it follows that A_{2m} and $2A_m$ have the same minimal polynomial.

Problem 2019-1/C (proposed by Nicky Hekster)

Let n be a prime number. Show that there are no groups with exactly n elements of order n. What happens with this statement if n is *not* a prime number?

Solution Solutions were submitted by Hans Samuels Brusse, Hendrik Reuvers and Paul Hutschemakers. The solution below is based on the solution by Hans.

Suppose G is a group with exactly n elements of prime order n. Let g be a group element of G of order n. Then $H_1 = \{1, g, g^2, \ldots, g^{n-1}\}$ is the subgroup generated by g and all $n-1$ elements $g, g^2, \ldots, g^{n-1}$ have order n. Since n is prime any of these elements can serve as generator for H_1.

Since G contains n different elements of order n by assumption, there must be exactly one more. Assume h is this last element, then h is not in H_1 and it will generate a different subgroup $H_2 = \{1, h, h^2, \ldots, h^{n-1}\}$. Note that $h, h^2, \ldots, h^{n-1}$ are distinct and do not belong to H_1, since this would imply $h \in H_1$. This gives us $2(n-1) \geq n$ distinct elements of order n, which leads to a contradiction unless $n = 2$.

In the case $n = 2$, we have distinct elements g, h of order 2. Note that ghg^{-1} has order 2 as well. Clearly, it cannot equal g, so it must equal h. However, this means g and h commute, and we find that the element gh is of order two and not equal to either g or h. So we find a contradiction in this case as well.

If n is not prime, the statement is false. For example, the abelian group $C_4 \times C_2$ (with C_k the cyclic group of order k) contains four elements of order four.

In the paper 'Finite groups that have exactly n elements of order n' by Carrie E. Finch, Richard M. Foote, Lenny Jones and Donald Spickler, Jr., *Mathematics Magazine* 75(3) (June 2002), pp. 215–219, the *finite* groups with the mentioned property are classified.