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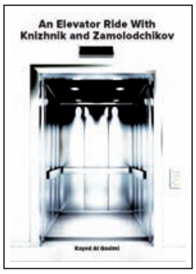

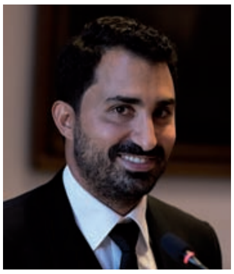

An Elevator Ride With Knizhnik and Zamolodchikov

Kayed Al Qasimi

On the 18th of April 2019 Kayed Al Qasimi from the University of Amsterdam successfully defended his PhD thesis with title *An Elevator Ride With Knizhnik and Zamolodchikov*. Kayed carried out his research under the supervision of Prof. dr. Jasper Stokman from the Korteweg-de Vries Institute of Mathematics and Prof. dr. Bernard Nienhuis from the Institute of Theoretical Physics.

The elevator ride

When we want to study physical systems we rely on measurements of physical quantities. The physical quantities that can be measured are called *observables*. Observables are like buildings, they often contain multiple floors which you need to visit in order to have a global view of the whole building. And buildings are always part of some city!

I spent my time in the city called *loop models* and all the attention was focused on two buildings on a particular street. These observables, the two buildings I mentioned, are called the *current* and the *nesting number*. However when you arrive at a building for the first time you have to take the stairs, and in the buildings I was interested in taking the stairs is very exhausting and time consuming. What we need is an elevator, a master elevator that would allow anyone to reach any floor at the push of a button. These elevators need to be built, but it is no simple task. Building such a master elevator was my ultimate goal. I could only achieve my goal when I realized that the city my buildings were located in was part of a larger world, this world is called the *skein category of the annulus*. Using the structure of this world and a blueprint for building elevators I was able to achieve my result and finally build the master elevator. The blueprint bears the names of two Russian physicists, Vadim Knizhnik and Alexander Zamolodchikov!

On the boundary of mathematics and physics

One of the joys of conducting research in mathematical physics is the discovery of new connections between different branches of mathematics and physics. Doing a mathematics and physics PhD (having both a mathematics and physics supervisor) I learned that mathematicians and physicists speak different languages. A lot of energy was spent just in translating the language and trying to learn to speak to my supervisors. While one (the mathematician) wanted precision and rigor, the other (the physicist) would ask why I was making things so complicated.

The city – Loop models

Loop models have emerged from the study of phase transitions and critical phenomena in statistical mechanics. They are a class

Pas gepromoveerden brengen hun werk onder de aandacht. Heeft u tips voor deze rubriek of bent u zelf pas gepromoveerd? Laat het weten aan onze redacteur.

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Figure 1 Local configurations.

of two-dimensional models defined on a lattice. The states of the models consist of continuous paths in space, that is the lattice, which do not end, except possible at the boundary. Paths may or may not intersect or overlap. A weight τ is also assigned to each loop. Configurations of the lattice are build up from the simple local configurations as shown in Figure 1.

Loop models have applications to τ-vector models, also known as $O(\tau)$ models. These are models where spins are unit vectors with τ-components. The name $O(\tau)$ refers to the fact that the spins have the symmetry of the orthogonal group. In physics, loop models have applications to condensed matter, can be used to model polymers in a bath and have connections to conformal field theory.

We were interested in the *dense loop model* with loop weight equal to one, on an $n \times \infty$ square lattice with periodic boundaries so that the lattice lies on the surface of an infinite cylinder with a circumference n. In particular we prove closed form expressions for two observables; the *current* and the *nesting number*. The current is the mean value that a specific type of path crosses a particular edge and the nesting number gives the probability that a point on the lattice is surrounded by a number of loops.

Proving the expressions for the observables amounts to proving recursion relations that connect the loop model of different sizes. One particular connection is the braid recursion which connects the model of lattice width n and $n+1$. Studying the braid recursion has led to a new connection between loop models, skein theory and quantum Knizhnik–Zamolodchikov (qKZ) equations. To understand the braid recursion I use skein theory to reinterpret the configurations of the model so that paths can cross each other. The result is the link-pattern tower, which describes how to connect configurations associated to the model of size n and $n+1$. To prove the braid recursion we show that the ground state of the model satisfies the qKZ equations and forms a qKZ tower of solutions. Constructing qKZ towers is though non-trivial (and existence is not guaranteed).

The closed form expression of the nesting number is a generating function for the probability of having a number of loops around a point on the lattice. A closed form expression means expressing it in terms of special functions (such as Schur functions or matrix determinants) that are indexed by the system size of the model. The ground state of the dense loop model is a function that takes values in some vector space v_n. I show it is a qKZ tower of solutions. This proves that the ground state satisfies the braid recursion. It describes how two solutions of different rank (n and $n+1$) are connected. In the solution of higher rank, that is $n+1$, we set the last variable equal to zero and we view this as a decent of the solution. For the solution of smaller rank, that is n we lift the representation space using an intertwiner. This is viewed as lifting the solution. The lift and descent connect the solutions giving the braid recursion.

The world – Skein theory

Skein theory originated from knot theory, where knot theory is an area of topology that studies knots. To study knots we use 2-dimensional diagrams. These are obtained by projecting the knot onto the plane $(x,y,0) \subset \mathbb{R}^3$ while keeping the information about the crossings appearing in the knot. An important step is to be able to distinguish and classify knots. One approach is to construct knot invariants.

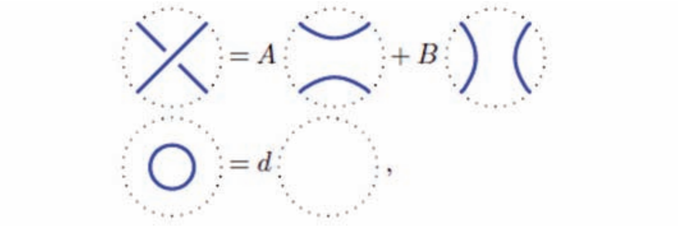

Figure 2 Kauffman rules.

Louis Kauffman introduced the bracket polynomial which allows to construct knot invariants based on elementary combinatorial rules on knot diagrams in which crossings are replaced by their two possible smoothings. The rules are often depicted as in Figure 2, where the disc shows the local neighborhood where the diagrams differ. The bracket polynomial $\langle K \rangle$ of a knot diagram K is obtained by recursively applying the two rules above to the knot diagram and is invariant under Reidemeister moves R2 and R3, see Figure 3, if $B = A^{-}$ and $d = -A^2 - A^{-2}$. Under this specialization the relations are called *skein relations*. Moreover, the bracket satisfies R1 up to a scalar. Multiplying the bracket polynomial by an appropriate factor involving the writhe of oriented knot diagrams one obtains an oriented knot invariant, which is the Jones polynomial.

In general, skein theory is the study of knots in 3-manifolds modulo the Kauffman skein relation and the loop removal relation. We deal with the 3-manifold $A \times [0,1]$, with A the annulus in the complex plane; the resulting geometry is a thickened cylinder.

This is the world in which our city, loop models, is located. And in the city, on some street, the two buildings, the current and the nesting number have been equipped with a master elevator allowing us to reach any floor with a click!

After completing the PhD project Kayed is planning to return to the United Arab Emirates with his family and start a career. Leaving the studying life and returning to the non-academic world will be a challenge, we wish the best to Kayed! At least in the UAE there will be no more winters!

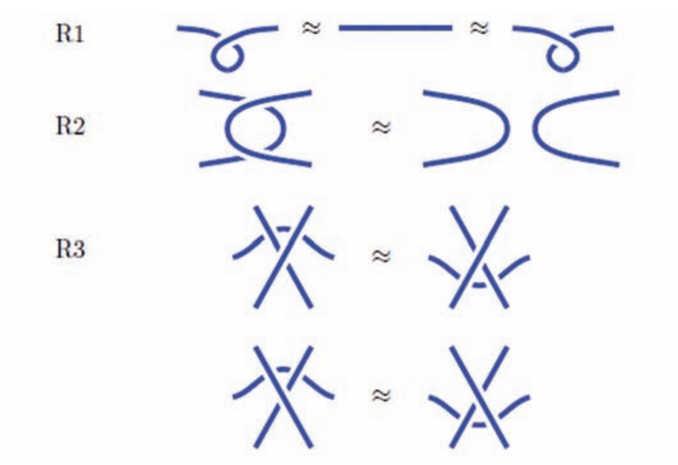

Figure 3 The Reidemeister moves.