Jean-Paul Allouche
CNRS, IMJ-PRG
Sorbonne Université, Paris
jean-paul.allouche@imj-prg.fr

Trip to the Proof

How I discovered an 'automatic' transcendence proof of the formal power series Pi

The formal power series π_C which can be seen as an analog of $\pi = 3.14159\ldots$ in positive characteristic was introduced by Carlitz and proved transcendental by Wade. Jean-Paul Allouche looks back upon his discovery of an 'automatic' transcendence proof (i.e., a transcendence proof using finite automata) of π_C in 1990.

In 1988, a time where authors of papers received after publication 'offprints' or 'reprints', i.e., bulk reproductions of their articles, I received a bunch of reprints of a paper at the *Compte-Rendus de l'Académie des Sciences*, about continued fraction expansions of algebraic formal power series in positive characteristic [2]. Among my reprints was, by mistake, a reprint of the article that followed my own article in the journal [18]. Its title, which I found intriguing, was 'Propriétés de transcendance des valeurs de la fonction zêta de Carlitz'. I looked at the paper, and was attracted by the following definition:

$$\pi := \prod_{n=1}^{\infty}\left(1-\frac{t^{q^n}-1}{t^{q^{n+1}}-1}\right) \quad \text{(sic)}$$

and by the claim that this number is transcendental. M. Mendès France easily convinced me that this number is algebraic! I will describe how this extra reprint and this misprint teased me and lead me to write the article [3].

More on the misprint

Let $\mathbb{F}_q$ the finite field with $q = p^a$ elements. Let $\mathbb{F}_q((1/t))$ be the field of Laurent series in $1/t$, i.e., the set of formal series $\sum_{n \geq n_0} a_n t^{-n}$ where a_n belongs to $\mathbb{F}_q$. Let π_C^* (C for Carlitz) denote the product π given above. Expanding it as a formal series in $1/t$, we see that π_C^* is in $\mathbb{F}_q((1/t))$. It is easy to prove that π_C^* is algebraic over $\mathbb{F}_q(t)$ (note that $\mathbb{F}_q(1/t) = \mathbb{F}_q(t)$). Namely, using the fact that the Frobenius map $y \to y^p$ is a morphism for addition and multiplication, we have

$$(\pi_C^*)^q = \left(\prod_{n=1}^{\infty}\left(1-\frac{t^{q^n}-1}{t^{q^{n+1}}-1}\right)\right)^q = \prod_{n=1}^{\infty}\left(1-\frac{t^{q^{n+1}}-1}{t^{q^{n+2}}-1}\right)$$

hence

$$\left(1-\frac{t^q-1}{t^{q^2}-1}\right)(\pi_C^*)^q = \pi_C^*$$

which proves that π_C^* is algebraic over $\mathbb{F}_q(t)$. Thus there must be a misprint in [18] in the definition of π, which is claimed to be transcendental.

Algebraicity in positive characteristics

It often happens in mathematics that slightly changing the formulation or the context of a possibly difficult problem leads to another problem which can be less difficult, or interesting, or both. It also can happen that the new problem does not throw any light on the one we started from. An example is to determine the transcendence (on the rationals) of a real number. How is this question modified if the numbers are written in base q and you perform addition and multiplication *without carries*? To write things more formally, take $q = p^a$ a prime power, let $\mathbb{F}_q$, $\mathbb{F}_q[t]$, $\mathbb{F}_q(t)$, $\mathbb{F}_q[[t^{-1}]]$, and $\mathbb{F}_q((t^{-1}))$ denote respectively the field with q elements, the ring of polynomials over $\mathbb{F}_q$, the field of rational functions with

characteristic 0	positive characteristic
$\mathbb{Z}$	$\mathbb{F}_q[t]$
$\mathbb{Q}$	$\mathbb{F}_q(t)$
$\mathbb{R}$	$\mathbb{F}_q((t^{-1}))$
$\cup$	$\cup$
{algebraic (over $\mathbb{Q}$) real numbers}	{algebraic (over $\mathbb{F}_q(t)$) Laurent power series}

Table 1

coefficients in $\mathbb{F}_q$, the ring of formal power series in $1/t$ over $\mathbb{F}_q$, i.e., the ring of formal power series $\sum_{n\geq 0} a_n t^{-n}$ with coefficients in $\mathbb{F}_q$ and the usual rules for adding and multiplying formal power series on some field, and the field of Laurent series in $1/t$, i.e., of Laurent series $\sum_{n\geq n_0} a_n t^{-n}$. Then, we have the analogy with the real numbers shown in Table 1.

One of my interests, going back to the eighties, is the result of Christol [16] and of Christol, Kamae, Mendès France and Rauzy [17] that gives a combinatorial characterization of the algebraic formal power series in positive characteristic.

Theorem 1 [16,17]. *Let $(a_n)_{n\geq 0} \in \mathbb{F}_q^{\mathbb{N}}$. Then the formal power series $\sum_{n\geq 0} a_n t^{-n}$ is algebraic over $\mathbb{F}_q(t)$ if and only if the set of subsequences $\{(a_{q^k n+r})_{n\geq 0}, k\geq 0, r\in[0,q^k-1]\}$ is finite.*

A sequence $(a_n)_{n\geq 0}$ such that the set of its subsequences (called the q-kernel of the sequence) $\{(a_{q^k n+r})_{n\geq 0}, k\geq 0, r\in[0,q^k-1]\}$ is finite, is called a q-automatic sequence. For more on q-automatic sequences, the reader can consult [7,19,21]. Contrarily to what is conjectured in the real case (typically that irrational algebraic numbers have a random-like base 10 expansion), Theorem 1 asserts that the coefficients of an algebraic formal power series in positive characteristic have some regularity. It is thus very tempting to play with this theorem to try discovering transcendental 'simple' formal power series. For example, I proved in [1] that the series $\sum s_2(n^2)t^{-n}$ is transcendental over $\mathbb{F}_2(t)$, where $s_2(n)$ is the sum modulo 2 of the binary digits of the integer n. But such series might be considered as *ad hoc* in that they do not occur 'naturally' in the literature, in particular in the literature published before 1979.

An allusion to the Carlitz functions

After reading the extra reprint [18], I could not resist to (I *had to*) read the papers of Carlitz and of Wade, and to learn what was going on. Carlitz introduced analogs of the exponential function, of the logarithm, of the Riemann zeta function, and of π. The right definition of Carlitz's π is actually

$$\pi_C := \prod_{n=1}^{\infty}\left(1-\frac{t^{q^n}-t}{t^{q^{n+1}}-t}\right).$$

Furthermore there is a mysterious relation between π_C, the Carlitz exponential function and the Carlitz zeta function that resembles the fact that $2i\pi$ is the period of the exponential function, while π^{2k} is a rational multiple of $\zeta(2k)$, where k is a positive integer and ζ the Riemann zeta function. Namely the period of the Carlitz exponential is equal to $(t^q-t)^{1/(q-1)}\pi_C$, while, letting for n a positive integer $\zeta_C(n) = \sum_{P \text{ monic} \in \mathbb{F}_q[t]} \frac{1}{P^n}$ denote the Carlitz zeta function, it can be proved that, for $(q-1)|n$, $\zeta_C(n)$ is a rational multiple of π_C (see [15], where Carlitz called ξ_∞ what is called π_C here). The reader will have noted that the fact of being even for the integer argument of the Riemann zeta function 'corresponding' to the fact of being divisible by $q-1$ for the argument of the Carlitz zeta function has something to do with the number of invertible elements of $\mathbb{Z}$ and of $\mathbb{F}_q[t]$: this can be used to complete Table 1.

A naive but fruitful approach

Comparing the 'true' value and the misprinted value of π_C made me think of a rather frequent approach of mathematical problems that I already alluded to above: if you do not really know how to solve a problem, find or invent a 'similar' problem and attack it. It might be the case that solving the latter helps to solve the former. But it might also be the case that you can combine these 'similar' questions and obtain a new question which might shed some light on the initial question. Here, given the form of the quantities π_C and π_C^* it is tempting to consider their quotient. An easy computation yields

$$\begin{aligned}\frac{\pi_C^*}{\pi_C} &= \prod_{n\geq 1}\left(1-\frac{t-1}{t^{q^{n+1}}-1}\right)\\ &= \prod_{n\geq 1}\left(1-\frac{t-1}{(t-1)^{q^{n+1}}}\right)\\ &= \prod_{n\geq 1}\left(1-\frac{1}{(t-1)^{q^{n+1}-1}}\right).\end{aligned}$$

To prove that π_C is transcendental, it suffices to prove that $\frac{\pi_C^*}{\pi_C}$ is transcendental, since π_C^* is algebraic. Furthermore the expression above is transcendental if and only if the same expression where $(t-1)$ is replaced by t is transcendental. Now why bother with π_C^*? Would it not be easier to introduce π_C^{**} defined by (compare with π_C^*)

$$\pi_C^{**} := \prod_{n=1}^{\infty}\left(1-\frac{t^{q^n}}{t^{q^{n+1}}}\right) = \prod_{n=1}^{\infty}\left(1-\frac{1}{t^{q^{n+1}-q^n}}\right)?$$

It is quite easy to prove that π_C^{**} is also algebraic, and that the quotient π_C^{**}/π_C is equal to

$$\frac{\pi_C^{**}}{\pi_C} = \prod_{n\geq 1}\left(1-\frac{1}{t^{q^{n+1}-1}}\right).$$

So that it is enough to prove the transcendence of the product P defined by

$$P := \prod_{n\geq 1}\left(1-\frac{1}{t^{q^n-1}}\right).$$

The next remark is a combinatorial one: P can be expanded as

$$P = \prod_{n\geq 1}\left(1-\frac{1}{t^{q^n-1}}\right) = \sum_{n\geq 1} a_q(n)t^n$$

where $a_q(n)$ is equal to 0 if n cannot be written as $n=\sum_{j\in J}(q^j-1)$ for some finite set of indices J, and $a_q(n) = (-1)^{\#J}$ if n can be written as $n=\sum_{j\in J}(q^j-1)$ (such a decomposition is unique if it exists). Thus, using the Christol theorem, to prove that P is transcendental, it suffices to prove that the sequence $(a_q(n))_{n\geq 1}$ is not q-automatic. Since the image of a q-automatic sequence is also q-automatic, it suffices to prove that the sequence $(b_q(n))_{n\geq 1}$ is not q-automatic, where $b_q(n) = |a_q(n)|$: in other words $(b_q(n))_{n\geq 1}$ is the characteristic sequence of the set of integers W defined by

$$\begin{aligned}\mathcal{W}_q := \{n\in\mathbb{N}, n = \textstyle\sum_{k\geq 0}\varepsilon_k(q^k-1),\\ \text{where } \varepsilon_k = 0 \text{ or } 1\\ \text{and } \varepsilon_k = 0 \text{ for } k \text{ large enough}\}.\end{aligned}$$

An unexpected coincidence

Once the interest of the set $\mathcal{W}_q$ above unveiled, I recognized $\mathcal{W}_2$ that Bétréma, Shallit and myself encountered in a quite different context [6]. Namely the von Neumann definition of integers (see, e.g., [14]) introduces sets $A_0, A_1, \ldots, A_n, \ldots$ defined by

$$\begin{aligned}A_0 &= \emptyset,\\ A_1 &= \{\emptyset\},\\ A_2 &= \{\emptyset, \{\emptyset\}\},\\ &\vdots\\ A_{n+1} &= A_n \cup \{A_n\},\\ &\vdots\end{aligned}$$

Replacing $\emptyset$ with $\{\}$, we get

$$\begin{aligned} A_0 &= \{\}, \\ A_1 &= \{\{\}\}, \\ A_2 &= \{\{\},\{\{\}\}\}, \\ &\vdots \\ A_{n+1} &= A_n \cup \{A_n\}, \\ &\vdots \end{aligned}$$

Hence, replacing the braces $\{$ with a and $\}$ with b, and suppressing the comma we obtain

$$\begin{aligned} A_0 &= a\ b, \\ A_1 &= a\ a\ b\ b, \\ A_2 &= a\ a\ b\ a\ a\ b\ b\ b, \\ &\vdots \end{aligned}$$

one can prove that the same sequence of 'words' on $\{a,b\}$, up to the final b can be obtained by iterating the morphism of monoid $a \to aab$, $b \to b$: start from a, and at each step replace simultaneously each a with aab and each b with b, so that, starting from a, yields

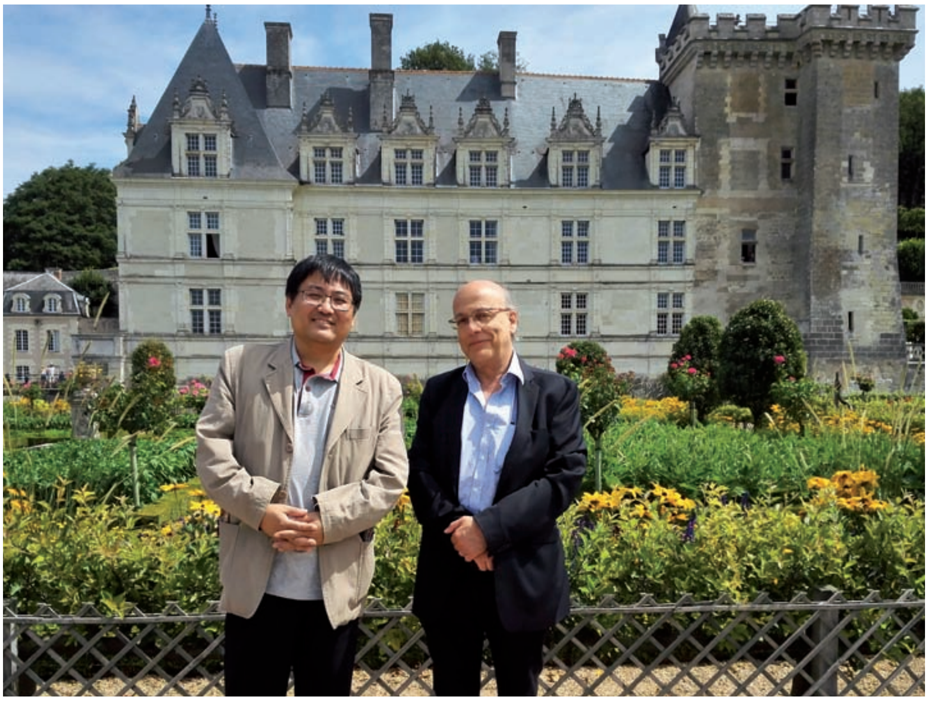
$$\begin{aligned} &a, \\ &a\ a\ b, \\ &a\ a\ b\ a\ a\ b\ b, \\ &\vdots \end{aligned}$$

The characteristic function of the indices of a's, namely the characteristic function of the sequence is nothing but $(b_2(n))_{n\geq 1}$, the characteristic function of the set

$$\begin{aligned} \mathcal{W}_2 := \{n \in \mathbb{N},\ n = \textstyle\sum_{k\geq 0} \varepsilon_k (2^k - 1),& \\ \text{where } \varepsilon_k = 0 \text{ or } 1& \\ \text{and } \varepsilon_k = 0 \text{ for } k \text{ large enough}\}.& \end{aligned}$$

We proved in [6] that the sequence $(b_2(n))_{n\geq 1}$ is not 2-automatic by using an idea of Shallit consisting of looking at the sequences $(b_2(2^k n + 2^k - k))_{n\geq 0}$, with $k \geq 2$. Namely we proved that $b_2(2^k n + 2^k - k) = 0$ for $n \in [0, 2^k - 3]$ and $b_2(2^k n + 2^k - k) = 1$ for $n = 2^k - 2$. This implies that all the sequences $(b_2(2^k n + 2^k - k))_{n\geq 0}$ are distinct, thus providing an infinite set of subsequences of b_2 of the form $(b_2(2^k n + j))_{n\geq 0}$ with $j \in [0, 2^k - 1)$. Hence b_2 is not 2-automatic since its 2-kernel is infinite, which implies that P is not algebraic using Theorem 1.

The proof for b_q is 'essentially' the same in that it 'suffices to make $2 = q$' (which is of course more difficult than to make $q = 2$), i.e., to try to generalize the proof for $q = 2$ by guessing what the right generalization of each step might be.



Jean-Paul Allouche (right) with Jia-Yan Yao (left)

This was not the end of the story

A link between π_C and the bracket series
In the same paper [3] we proved via Christol's theorem another transcendence result already proved by Wade in [29].

Theorem 2 (Wade). *The bracket series $\sum_{k\geq 1} \frac{1}{[k]}$ is transcendental over $\mathbb{F}_q(t)$, where $[k] := t^{q^k} - t$.*

Our proof was 'simple' in that it used only the Christol theorem and the fact that for any integer $M > 0$ there exist infinitely many primes congruent to 1 modulo M (recall that this last claim can be proved by elementary methods without using the whole Dirichlet theorem on primes in arithmetic progressions, see, e.g., [25]).

It happens, as we noted later on in [3], that $\frac{\pi'_C}{\pi_C} = \sum_{k\geq 1} \frac{1}{[k]}$, where π'_C is the derivative of π_C with respect to t. Since, as the reader can easily prove, the derivative of an algebraic formal power series is also algebraic, this gives that the transcendence of the bracket series implies the transcendence of π_C, yielding another simple proof of the transcendence of π_C.

More in this direction. A dream
A year later V. Berthé proved what the author did not succeed in proving, namely that Christol's theorem can be used to prove that, letting $\zeta_C(k)$ denote the Carlitz zeta function, the quotient $\zeta_C(k)/\pi_C^k$ is transcendental for $k \in [1, q-2]$ (see [10, 11]). Actually the quantities $\zeta_C(k)$ for any $k \geq 1$ and $\zeta_C(k)/\pi_C^k$ for any k not divisible by $q-1$ are transcendental: this was proved by J. Yu in 1991 [34], using the deeper and more complicated theory of Drinfeld modules. Several papers were then devoted to proving transcendence (or algebraicity) results for functions à la Carlitz in positive characteristic (see, e.g., [5, 8, 9, 12, 13, 20, 22, 23, 24, 26, 28, 30, 31]; also see the survey [27]).

At that time, I had (and I still have) a kind of dream, wishing that the various methods used to prove the transcendence of quantities à la Carlitz could be 'unified'. There were four methods: the Wade method, the criteria à la De Mathan, the 'automatic method', and last but not least the powerful Drinfeld modules. This wish was only partially realized: the first three methods were (essentially) unified. The readers can consult the papers [20, 28, 32, 33]: they might then want to attack the question of unifying these three methods with Drinfeld modules...

Acknowledgments
We warmly thank Jia-Yan Yao for his help in pointing out several references and R. Fokkink for several very useful remarks.

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