Problem

| Problem Section

This Problem Section is open to everyone; everybody is encouraged to send in solutions and propose problems. Group contributions are welcome. For each problem, the most elegant correct solution will be rewarded with a book token worth € 20. (To compete for the book token you should have a postal address in The Netherlands.)
Please send your submission by e-mail (LaTeX is preferred), including your name and address to problems@nieuwarchief.nl.
The deadline for solutions to the problems in this edition is 1 March 2018.

Problem A

Consider two identical bags of stones, all having integral weights. No two weights in a bag are the same. Suppose that the stones from these two bags are divided into proper subsets of equal cardinality and equal total weight. No two weights in a subset are the same. Is it then possible to readjust this division such that each subset contains stones from the same bag?

Problem B

Let $H_1,\ldots,H_k$ be k planes in $\mathbb{R}^3$. Prove that the unit ball contains an open ball of radius $\frac{1}{k+1}$ which does not intersect any of the planes.

Problem C (proposed by Hendrik Lenstra)

Suppose that a,b,n,m are integers, and that m,n are positive, such that $a^n \equiv b^n \equiv -1 \bmod m$. Prove there exists an integer c such that $ab \equiv c^2 \bmod m$. Also provide a fast method to compute such a c which works even if the prime factors of m are unknown.

Edition 2017-2 We received solutions from Rik Bos (Bunschoten), Souvik Dey (Kolkata, India), Math Dicker (Hoensbroek), Alex Heinis (Amsterdam), Thijmen Krebs (Nootdorp), Hans van Luipen (Zaltbommel), José Polo Gómez (Sevilla, Spain), Ludo Pulles (Zeist), Hendrik Reuvers (Maastricht), Hans Samuels Brusse (Den Haag), Toshihiro Shimizu (Kawasaki, Japan), Djurre Tijsma (Zeist) and Robert van der Waall (Huizen). The book tokens for problems A, B and C go to Ludo Pulles, Thijmen Krebs, repectivily Rick Bos.

Problem 2017-2/A

Show that there are no infinite antichains for the partial order $\leq$ on $\mathbb{N}^k$ defined by $(x_1,x_2,\ldots,x_k) \leq (y_1,y_2,\ldots,y_k)$ iff $x_i \leq y_i$ for all $1 \leq i \leq k$.

Solution This problem was solved by Alex Heinis, Thijmen Krebs, Hans van Luipen, Ludo Pulles, Hendrik Reuvers, Hans Samuels Brusse, Toshihiro Shimizu and Djurre Tijsma. Several solvers noticed that this is Dickson's Lemma. All solutions are similar and Thijmen Krebs is very concise. Below is his solution.
By induction. The statement is obviously true if $k=0$. Suppose on the contrary that such an antichain C exists, and let $x \in C$. For every other $y \in C$ there is some $1 \leq i \leq k$ such that $0 \leq y_i \leq x_i$. By the pigeonhole principle we find some infinite antichain $A \subseteq C$ and some $1 \leq i \leq k$ such that $y_i = z_i$ for all $y,z \in A$. Deleting the i-th entry from each $y \in A$ then gives an infinite antichain in $\mathbb{N}^{k-1}$, which is impossible by induction.

Problem 2017-2/B

A sector is a portion of a disk enclosed by two radii and an arc. For each pair of radii, there are two complementary sectors. If the two complementary sectors have unequal area, then

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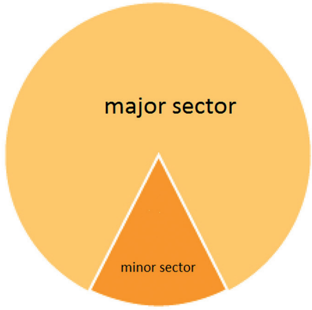
we say that the larger sector is the major sector. Let S be a subset of the plane. We say that $x \in S$ is virtually isolated if x is the only element of S in a major sector of a disk centered on x. Suppose that all elements of S are virtually isolated. Prove that S is countable.



Solution The problem was suggested by Jan Aarts, with a solution. The problem was solved by Thijmen Krebs, José Polo Gómez and Toshihiro Shimizu. Below is the solution of Thijmen Krebs. Again very concise.
Each $x \in S$ is the center of some closed semi-disk with rational radius and rational angle that contains no point from S other than x. Distinct $x, y \in S$ with up to translation the same semi-disk must be more than its radius apart, so sending $x \in S$ to a translation of its semi-disk to a rational center within half the radius from x is an injection to a countable set.

Problem 2017-2/C (proposed by Hendrik Lenstra)

Let n be a natural number > 1. Suppose that for every prime $p < n$ we have that $p^n = (p-1)^n + 1 \bmod n^2$. Prove that $n = 2$.

Solution Solved by Rik Bos, Souvik Dey, Math Dicker, José Polo Gómez, Thijmen Krebs, Hans van Luipen, Toshihiro Shimizu, Djurre Tijsma and Robert van der Waall. All solutions are similar. The solution by Rik Bos is below.
We first argue by induction that $a^n = a \bmod n^2$ for all $a < n$. If a is prime then $a^n = (a-1)^n + 1 \bmod n^2$, which by induction is equal to $(a-1) + 1 \bmod n^2$. If a is composite, then $a = bc$ and by induction both $b^n = b \bmod n^2$ and $c^n = c \bmod n^2$. Therefore, $a^n = a \bmod n^2$. By the binomial theorem $(n-1)^n = (-1)^n \bmod n^2$. Taking $a = n-1$ produces $(-1)^n = n - 1 \bmod n^2$. It follows that $n = 2$.