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KK-theory, gauge theory and topological phases

The history of science is full of moments when physics and mathematics both benefited from mutual interaction, with the early twentieth century providing us with two big examples. On one hand, Einstein's theory of gravitation could not have been developed without the work of Riemann on the geometry of manifolds. On the other hand, the advent of quantum mechanics fostered the development of new mathematics especially in operator algebras. In line with this spirit of interdisciplinarity, the school and workshop 'KK-theory, Gauge Theory and Topological Phases' took place from 27 February to 10 March 2017 at the Lorentz Center in Leiden. Francesca Arici and Domenico Monaco report about this event.

The program of 'KK-theory, Gauge Theory and Topological Phases', organized by Alan Carey (Canberra, AU), Steve Rosenberg (Boston, USA), and Walter van Suijlekom (Nijmegen, NL), aimed both at young and more senior scientists in mathematics and physics, focusing on the recent applications of Kasparov's bivariant K-theory in both high-energy and solid state physics.

This event was able to bring together experts from several diverse communities, creating an environment that stimulated fruitful interactions, fostering new interdisciplinary collaborations, and allowing participants to learn about cutting-edge research on the topics of the program from the leaders in the field.

The main subject of the program was *KK-theory*, an apparently abstract notion from operator algebra which has recently emerged as an important tool for applications of noncommutative geometry in mathematical physics, to describe gauge-theoretic phenomena originating both in high-energy physics and in condensed matter. Roughly speaking, KK-theory produces 'topological invariants' for C^*-algebras, which are a particular class of operator algebras often considered to be geometric: they include the algebras of continuous functions over (Hausdorff, locally compact) topological spaces, and generalize topological notions to a noncommutative algebraic setting. Besides, C^*-algebras have repeatedly proved to be useful in modelling physical systems as well, most notably in quantum mechanics, where Max Born's commutation relations $[x,p]=\mathrm{i}\hbar$ (which imply the paradigmatic result of quantum physics, namely Heisenberg's uncertainty principle) are considered to be the archetype of a noncommutative phase space.

Since its early days the field of noncommutative geometry and operator algebras has produced results of striking mathematical beauty, but also of profound relevance for mathematical physics. In the recent years, noncommutative index theory and KK-theory have provided insight in problems in particle physics [34], and in the mathematical formulation of the so-called 'bulk-edge correspondence' in solid state physics (see the subsection 'Topological phases' below). With these recent applications of KK-theory to gauge theories and topological quantum matter, Wigner's 'unreasonable effectiveness of mathematics' strikes once again.

The school

The first week of the program was devoted to the school. There were three main short courses, each of which was taught by two experts on the subject. The aim of these lectures was to get everyone 'up to speed' for the incoming workshop, and they served this purpose excellently, providing very clear exposition on the subjects of the program.

Each day of the school concluded with a discussion session, where the participants were invited to raise their doubts about the lectures, or to ask for clarifications regarding some of the points that were raised during them. This favoured a thorough understanding of the subjects taught, as well as the possibility to debate the physical motivations and applications for the mathematical tools that were illustrated and developed in the courses.

Index theory

The foundations of modern-day noncommutative geometry can be found in *index theory* [15]. Two introductory lectures on this subject were delivered by Matthias Lesch (Bonn, DE). The main aim was to state and discuss the celebrated *Atiyah–Singer index theorem* [3], one of the milestones of modern mathematics which draws a bridge between analysis (elliptic pseudo-differential operators on manifolds), topology (characteristic classes of vector bundles) and operator algebra (the theory of Fredholm operators). For this result Sir Michael F. Atiyah got both the Fields Medal (in 1966) and the

KK-theory

Kasparov's bivariant K-theory [21], or KK-theory, is a framework which incorporates and generalizes both K-theory and K-homology for C^*-algebras. In the spirit of noncommutative geometry, it can be thought of as an extension of index theory in which smooth closed manifolds are replaced by singular spaces, in the form of C^*-algebras.

Elements of the group $KK(A,B)$, also called *Kasparov modules*, are (homotopy classes of) triples (E,ϕ,T), where

1. E is a ($\mathbb{Z}_2$-graded) right Hilbert B-module,
2. ϕ is a representation of A on E as adjointable operators, and
3. T is an adjointable operator on E (of odd degree) such that the operators $[T,\phi(a)]$, $(T^*-T)\phi(a)$ and $(T^2-\mathbf{1})\phi(a)$ are compact for all $a \in A$.

One can show that $KK(\mathbb{C},B)$ reproduces $K_0(B)$, the usual K-group of the C^*-algebra B, while $KK(A,\mathbb{C})$ gives back the K-homology group of the algebra A, that is, the group of K-cycles (or Fredholm modules) over A.

One of the deepest and most powerful features of Kasparov's bivariant K-theory is the fact that KK-cycles can be composed: one can define the internal *Kasparov product*

$$KK(A,B)\times KK(B,C)\to KK(A,C).$$

In the case $A=C=\mathbb{C}$, the internal Kasparov product reduces to the well-known *index pairing* of K-theory classes on B with Fredholm modules, landing in $KK(\mathbb{C},\mathbb{C})\simeq\mathbb{Z}$. Thus, one recovers classical index theory within the framework of KK-theory.

Motivated by the appearance of Dirac type operators in the index pairing, an unbounded version of KK-theory has also been formulated [4], where instead of an adjointable operator one considers a densely defined, self-adjoint and regular operator T on the Hilbert B-module E, satisfying the following two conditions:

1. $\phi(a)(T+i\mathbf{1})^{-1}$ is a compact operator on E for all $a\in A$, and
2. there exists a dense $*$-subalgebra $\mathcal{A}\subset A$ such that for all $a\in\mathcal{A}$ the operator $\phi(a)$ leaves the domain D of T invariant, and moreover the (graded) commutator $[T,\phi(a)]$, initially defined on D, admits a bounded extension to the whole E.

In particular, unbounded representatives of K-cycles — i.e. elements in $KK(A,\mathbb{C})$ — are spectral triples $(\mathcal{A},\mathcal{H},D)$, where $\mathcal{A}$ is a dense subalgebra of A, whose elements have bounded commutators with D.

The two pictures are related via the bounded transform: given an unbounded Kasparov module, the formula

$$F_D := D(1+D^2)^{-1/2}$$

gives a bounded (A,B)-Kasparov module. Note that the bounded transform is what allows one to move from Dirac operators to Fredholm operators.

While the unbounded theory presents more complication due to analytic subtleties, it also offers several advantages because it is more explicit and more geometric. Moreover, it allows to give, up to equivalence, an explicit realisation of the Kasparov product. This constructive approach to the Kasparov product, initiated by Mesland [26] and rigorously developed by Mesland, Kaad, Lesch, Rennie and others [19,27,28], has opened the door to a wide range of applications of geometric and physical nature. In the wake of these developments come new applications of Kasparov theory to index theory, with new proofs of many classical theorems by KK-methods.

The poster for the school and workshop depicts the complex Hopf fibration, an example of geometry that can be factorised using the Kasparov product.

Abel Prize, together with Isadore Singer (in 2004).

Fredholm operators are characterized by the property of having finite-dimensional kernel and cokernel. Many examples of such operators come from the field of *pseudo-differential operators*, a class of operators between vector bundles over a manifold which, roughly speaking, act in the Fourier representation by multiplication times a function (the *symbol* of the operator) which is smooth and doesn't grow too fast at infinity. The Atiyah–Singer theorem computes the *index* of an *elliptic* pseudo-differential operator — a purely analytic concept which is defined, for any Fredholm operator, as the difference between the dimensions of its kernel and its cokernel — in terms of certain characteristic classes of the manifold and of the bundles on which the operator acts (on the 'topological' side). We refer to the popular article [24] for an introduction. At a more abstract level, the index can be realized

Gauge theory

In the mathematical approach to gauge theories, gauge fields are realized as connections on principal bundles up to equivalence under the action of the gauge group.

The Atiyah–Singer index theorem has proven to be an effective tool in understanding the mathematical aspects of gauge theories: in the study of four-dimensional Yang–Mills gauge theories, this theorem was used to determine the dimension of the moduli space of self-dual connections [2]. With this in mind, it is not surprising that KK-theory, which sets its roots in index theory, has important and promising applications in the field of gauge theories.

In the framework of noncommutative geometry, gauge theories naturally arise from spectral triples [11]. Indeed, out of the ingredients of a spectral triple $(\mathcal{A},\mathcal{H},D)$, one can naturally obtain the defining elements of a gauge theory.

- Starting from the algebra $\mathcal{A}$ one constructs the gauge group $\mathfrak{G}$, by looking at the group of inner automorphisms, which in the case of noncommutative algebras is non-trivial. In many situations, one can identify the group $\mathrm{Inn}(\mathcal{A})$ with the gauge group of the theory.
- The gauge fields are obtained from the spectral data contained in the Dirac operator, with its spectrum playing a central role in the theory.
- The gauge group $\mathfrak{G}$ acts on the Dirac operator (i.e., on fields) by conjugation with a unitary operator $D \mapsto UDU^*$, giving rise to a perturbation in terms of pure gauge fields. Since the spectrum of the Dirac operator is invariant under the action of unitary gauge transformations, one can define the *spectral action*, which is interpreted as an action functional for the theory, describing the dynamics and interactions of the gauge fields coming from D.

As described amply in [34], this notion is compatible with the classical definition of gauge theories in terms of connections on a vector bundle. Moreover, one of the greatest achievements of the noncommutative approach to gauge theories is the derivation of the full Standard Model of particle physics starting from a spectral triple on a noncommutative manifold which is the product of a commutative geometry and a matrix algebra (cf. [33]).

Motivated by this product structure, one looks at the unbounded version of KK-theory, which allows for a bundle-theoretic description of gauge theories, both on commutative and noncommutative base spaces. In [9] the constructive Kasparov product is used to factorize a class of noncommutative differential geometries, in the form of spectral triples, into the product of two pieces: a commutative horizontal spectral triple on the base manifold, and a Kasparov module which describes the vertical noncommutative geometry.

More generally, it is interesting to understand whether a given spectral triple factorizes as the product of an unbounded Kasparov bimodule with another spectral triple, on a possibly noncommutative manifold, which would then represent the base manifold on which the gauge theory is defined. Many amongst the participants and speakers of the program have been dealing with this question in the past years, and their work has shed light on the problem, as well as on the technical aspects of Kasparov's theory.

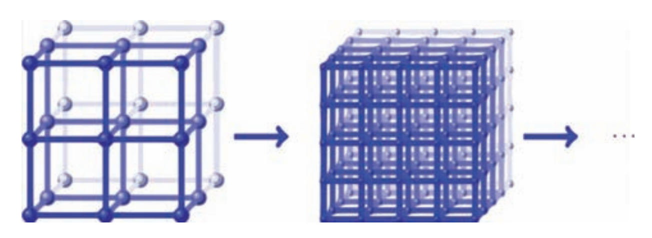


Illustration: Walter van Suijlekom

In lattice gauge theories, 'refining' the lattice yields the thermodynamic limit. This procedure is best described in terms of limits of C^*-algebras.

as a pairing between K-theory (vector bundles) and K-homology (elliptic differential operators).

In the paradigm of noncommutative geometry due to Connes [11, 12] a central notion is that of a *spectral triple* $(\mathcal{A},\mathcal{H},D)$. This consists of an involutive algebra of operators acting on a Hilbert space $\mathcal{H}$ and a self-adjoint operator D on the same Hilbert space, subject to certain conditions. Such Dirac operators are 'unbounded versions' of Fredholm operators, as described in the box on KK-theory. Commutative examples of spectral triples are provided by the Hodge–de Rham operator on differential forms over a Riemannian manifold, and the spin Dirac operator that is naturally defined when the manifold has also a spin structure, which is a structure slightly finer than orientability [25] used in physics to model the behaviour of particles like electrons. Conversely, the notion of spectral triple, as introduced by Alain Connes, is modeled on these examples, and generalizes them to the noncommutative realm.

Coming back to the program of the school, more advanced topics in index theory, related to *Kontsevich formality theorem*, were discussed instead by Ryszard Nest (Copenhagen, DK). These shed some light on the connection between index theory and *Poisson geometry*, and on how this link naturally leads to the study of *Hochschild cohomology* of manifolds. This inspired Connes to formulate his *cyclic cohomology* for spectral triples, which heuristically plays the role for a noncommutative manifold of the usual de Rham cohomology of differential forms.

The presentation of index theory by Lesch and Nest served as an essential 'motivating example' for the other lectures taught during the school.

KK-theory

Siegfried Echterhoff (Münster, DE) gave an introductory overview on the main protagonist of the program, namely KK-theory (see box 'KK-theory').

In its complex, bounded variant, this theory associates to pairs of ($\mathbb{Z}_2$-graded) C^*-algebras (A,B) an abelian group, denoted by $KK(A,B)$. In order to construct the KK-group, one needs to move beyond Hilbert spaces, the natural spaces on which C^*-algebras are represented as operators, and deal with more general modules, endowed with a scalar product taking values in a (possibly different)

Topological phases

The recent years have witnessed a booming expansion in the theoretical understanding and experimental realization of so-called *topological insulators* [10, 17], a class of materials which are insulating in the bulk but display robust edge modes, capable of conducting charge or spin without dissipation and virtually without being affected by the presence of impurities and defects in the system. Such topological materials, whose theoretical proposal has already lead to the assignment of several Nobel Prizes in Physics (including the latest one from 2016), could potentially be employed in the future in spintronic devices and in quantum computers.

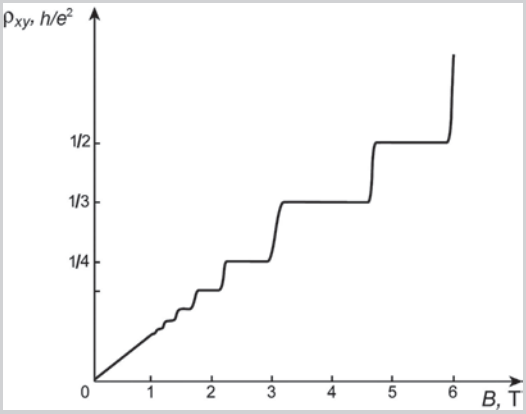
At a theoretical level, topological insulators are classified according to the *symmetries* satisfied by the Hamiltonian H that models an effective one-particle picture of the quantum system. The fundamental symmetries come in three flavours:

- *charge-conjugation* (also called *particle-hole*): it is implemented by an antiunitary operator $\mathcal{C}$ over the one-particle Hilbert space which can square to $\pm\mathbf{1}$, and H is said to be charge-conjugation symmetric if
$$\mathcal{C}H\mathcal{C}^{-1}=-H;$$
- *time-reversal*: it is also implemented by an antiunitary operator $\mathcal{T}$ squaring to $\pm\mathbf{1}$, and H is said to be time-reversal symmetric if
$$\mathcal{T}H\mathcal{T}^{-1}=H;$$
- *chiral*: this symmetry $\mathcal{S}$ is implemented unitarily, squares to $\mathbf{1}$, and H is said to be chiral symmetric if
$$\mathcal{S}H\mathcal{S}^{-1}=-H.$$

Owing to the fact that $\mathcal{C}$ and $\mathcal{T}$ can square either to $+\mathbf{1}$ or $-\mathbf{1}$, and that the combination $\mathcal{C}\mathcal{T}$ is a chiral symmetry $\mathcal{S}$, one is lead to ten distinct symmetry classes: two of them, the one with absence of any symmetry and the one where only a chiral symmetry is present, are dubbed *complex*, while the other eight, which have at least one antiunitary symmetry and hence need the specification of a complex conjugation operator, are instead called *real*. When the underlying system is a crystal, other (spatial) symmetries may be imposed, like invariance under translations of the crystalline Bravais lattice and possibly under reflections or discrete rotations.

To each of these symmetry classes, and depending on the dimensionality of the system, one can attach certain topological labels, which essentially encode how many inequivalent Hamiltonian there exist obeying the prescribed symmetries: two Hamiltonians are considered equivalent if they can be adiabatically deformed one into the other, that is, if there is an homotopy of Hamiltonians between them for which the bulk gap stays open. Some of these topological labels have actual physical meaning as measurable quantities: the most celebrated of these instances is the quantization of the *quantum Hall transverse conductivity*, which occurs in a 2-dimensional gapped periodic system where all the other symmetries above are broken (the so-called Class A), and which is driven by the integer Chern number associated to the occupied Bloch states (see [16] and references therein). However, some of these classes, like the one corresponding to time-reversal symmetric systems in dimensions 2 and 3 under the assumption that $\mathcal{T}^2=-\mathbf{1}$ (Class AII), have more peculiar features, like the appearance of *torsion invariants*, with values in $\mathbb{Z}_2$ [13, 14, 20].

This classification scheme has lead to the compilation of 'periodic tables' of topological insulators [23, 31], depicted in Table 1. The realization that these tables are indeed periodic is the first success of mathematics, and more specifically of K-theory, in this field: *Bott periodicity* can be applied to show how to relate symmetry classes in different dimensions and how the classification repeats itself with an 8-fold periodicity in the dimension [32].



The transverse resistivity exhibits plateaus in the quantum Hall effect. Index pairings in KK-theory lurk behind the curtains of this and many other quantization phenomena in condensed matter physics.

C^*-algebra. These objects are known as *Hilbert C^*-modules*. Much of the theory of Hilbert spaces and operators thereon carries through to this more general setting, but with notable exceptions: not all bounded operators posseses an adjoint, and the class of *adjointable* operators plays a fundamental role in the theory. In particular, elements in the KK-group $KK(A,B)$ are Hilbert B-modules on which A acts (that is, is represented) as adjointable operators.

Starting from a brief reminder on K-theory for C^*-algebras, Echterhoff presented the theory of Hilbert C^*-modules, of the classes of operators acting on them (e.g. adjointable and compact operators), and of C^*-correspondences (in particular Morita equivalences), concluding with the definition of KK-theory in its bounded version and with the construction of a representative for the *Kasparov product*.

Echterhoff's lectures were followed by the ones given by Jens Kaad (Odense, DK), which instead presented the *unbounded picture* of KK-theory. Much like in the case of spectral triples, in this framework the Fredholm operator which enters in the definition of a KK-class is replaced by a Dirac-like (unbounded) operator. This requires first of all to set up a theory for unbounded self-adjoint operators on Hilbert C^*-modules, which unfortunately combines the technical difficulties arising in the theory of unbounded operators on standard Hilbert spaces with those of operators on

C^*-modules, as for example the definition of the adjoint. Kaad gave an exposition of this theory, proceeding with the illustration of the main issues and possible solutions thereof for the definition of an unbounded representative of the Kasparov product (see box 'KK-theory'). The general theory was corroborated with several instructive examples, both from the commutative and the noncommutative perspective.

Topological phases

Index theory and K-theory have found applications also in the thriving field of *topological quantum matter* (see box 'Topological phases'). The seemingly trivial observation that the index of a Fredholm operator, being defined as a difference of dimensions of vector spaces, is an integer, has profound implications, underlying for example the quantization of the noncommutative topological invariants that label certain symmetry classes of topological insulators. These are a class of recently discovered materials, which offer promising applications in quantum devices due to their peculiar transport properties, induced mainly by their symmetry properties.

New advancements of the theory, which were discussed during the program, widely generalize the treatment of topological phases of matter to include *disordered systems*, *quasi-crystals* and *aperiodic solids* [29,30]. Inspired by the Bellissard–Connes approach to the quantum Hall effect [5], the correct framework to describe the symmetry classes of such systems was recently shown to be the KK-theory of certain C^*-algebras of observables, which are essentially constructed from the Hamiltonian of the system and the space of disorder or atomic configurations. Kasparov's K-theory has indeed proven to be a flexible tool to incorporate all the relevant classes of symmetries, by considering in particular real and Real C^*-algebras (in the sense of Atiyah [1]: Real C^*-algebras are noncommutative spaces endowed with an involution). Furthermore, the use of the Kasparov product allowed to prove certain mathematical versions of the *bulk-edge correspondence*, which relates the topology of the bulk Hamiltonian to an effective description of the boundary of the system when this is terminated by an edge. These results prove rigorously the defining property of topological insulators, namely that their 'topological' character reflects in the stability and robustness of their edge states even in the disordered regime [6,7,8,22].

Symmetry				Dimension							
AZ	$\mathcal{T}$	$\mathcal{C}$	$\mathcal{S}$	1	2	3	4	5	6	7	8
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

Table 1 The periodic table of topological insulators [23,31,32]. In the first column, 'AZ' stands for the Altland–Zirnbauer (sometimes called Cartan) label. The labels for the symmetries are: $\mathcal{T}$ (time-reversal), $\mathcal{C}$ (charge-conjugation), $\mathcal{S}$ (chirality). Time-reversal symmetry and charge conjugation are $\mathbb{Z}_2$-symmetries implemented antiunitarily, and hence can square to plus or minus the identity: this is the sign appearing in the respective columns (0 stands for a broken symmetry). Chirality is instead implemented unitarily: 0 and 1 stand for absent or present chiral symmetry, respectively. Notice that the composition of a time-reversal and a charge conjugation symmetry is of chiral type. The table repeats periodically after dimension 8 (i.e. for example the column corresponding to $d=9$ would be equal to the one corresponding to $d=1$, and so on).

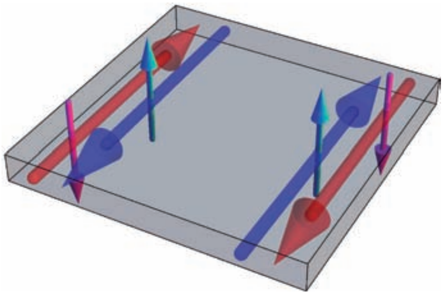
An introduction to the physics and mathematics of topological phases of matter was delivered by Guo Chuan Thiang (Adelaide, AU) and Chris Bourne (Sendai, JP).

Thiang introduced the periodic table of topological insulators (see Table 1), discussing symmetries of a quantum system from the point of view of K-theory and Clifford algebras. Indeed, the topology encoded in the bundle of occupied states of a gapped Hamiltonian determines a K-theory class of a C^*-algebra which essentially depends on the spatial symmetries (for example, a noncommutative torus in the case of periodic systems). Charge-conjugation, time-reversal and chiral symmetries form a representation of some Clifford algebra on the Hilbert space of the system, which turns the above-mentioned bundle into a Clifford module, and refines its geometry in terms of topological invariants. Thiang illustrated the general theory with two main examples from the periodic table, namely Class A in dimension 2 (the quantum Hall effect) and Class AII in dimensions 2 and 3 (the time-reversal symmetric topological insulators first proposed by Liang Fu, Charlie Kane and Eugene Mele [13,14]), which are paradigmatic for what concerns $\mathbb{Z}$-valued and $\mathbb{Z}_2$-valued invariants, respectively.

Bourne focused instead on the inclusion in this framework of disorder effects by means of C^*-algebraic techniques. This theory originated in the works of Jean Bellissard and collaborators, who gave a mathematical description of the quantum Hall effect based on the Connes–Chern character. The latter can be understood as a pairing between the K-theory class of the occupied states of the Hamiltonian (living in the K-theory of a *crossed product* C^*-algebra of the disorder configurations by the action of the group of translations by integer shifts) and a certain K-cycle (that is, a Fredholm module or a spectral triple), dictated essentially by the symmetries of the system. The fact that this pairing takes values in the group of integers is in turn a consequence of an index theorem.

This approach has been recently generalized to explain all the classes of the periodic table, as Bourne illustrated in his lectures. KK-theory arises as a description of the cycle needed to pair with the physical input from the Hamiltonian. Antiunitary symmetries like charge-conjugation and time-reversal require in particular certain 'reality' conditions, both in the algebra and in this KK-cycle, which in turn give rise to the torsion invariants, with values in $\mathbb{Z}_2$, via some index pairings in KK-theory.

Bourne's lectures also covered the so-called *bulk-edge correspondence*, a principle which allows to 'read' the topology of the bulk physical system from the effects it induces on boundary states. It is this principle that underlies, for example, the quantization of the Hall current on the boundary of a sample of a quantum Hall system. This effect can be understood in terms of Toepliz-like extensions of the 'bulk' algebra: the long-exact sequences in KK-theory then explain how to relate bulk and edge invariants.



Robust boundary currents and an insulating bulk are the defining properties of topological insulators. The bulk-edge correspondence has been established mathematically thanks to KK-theory methods.

The workshop

The five-days workshop followed the school and built on the topics introduced there. Elaborating on the material presented in the school, the newest uses of KK-theory in index theory, gauge theories and topological phases of quantum matter were illustrated by leading experts like Simon Brain (Nijmegen, NL), Maxim Braverman (Boston, USA), Motoko Kotani (Sendai, JP), Bram Mesland (Bonn, DE), and Emil Prodan (New York, USA). Together with several other international experts, the speakers reported on cutting-edge research, recently developed in the fields of interest to the program.

The more topical advancements and applications of KK-theory were debated both alongside the presentations and during dedicated discussion sessions, which served as well as a means to outline future lines of investigation for the communities which came together during this program.

The first discussion session was devoted to 'KK-theory and gauge theory' (see box 'Gauge theory'), chaired by Gianni Landi (Trieste, IT). The discussion concerned both open problems and applications of KK-theory, including the interpretation of KK-cycles as morphisms in the category of C^*-algebras, but also uniqueness of KK-theory and connections with the theory of extension of C^*-algebras. On the mathematical physics side, various applications were discussed, including the use of factorization in KK-theory to decompose space-time into hypersurfaces.

Ralph Meyer (Göttingen, DE) presided over a second discussion session focused on 'KK-theory and topological phases'. The interdisciplinary atmosphere of the conference aroused the curiosity of the mathematicians in the audience for the physical aspects of topological quantum matter. For example, the discussion dealt with the possibility of measuring the invariants produced by KK-theory in the lab, in the spirit of the observation of Chern number as quantum Hall conductivity. The challenge for experimentalists and theoreticians alike lies in particular in the measurement of bulk torsion invariants: the topology of $\mathbb{Z}_2$ insulators is usually probed through the edge states of the system.

The discussion sessions reflected several aspects of problems being currently investigated by participants in the workshop, and fostered both new and ongoing collaborations.

With its comfortable facilities and ample office space, the Lorentz Center proved to be a more than adequate accommodation for this kind of event, allowing and encouraging discussions and collaboration between the participants, and taking the 'bureaucratic' load off of the shoulders of the scientific organizers, under the motto "You focus on the science — we do the rest!"

Acknowledgement
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