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## Research

# Morley chains of osculant curves

**At the end of the nineteenth century the algebraic geometer Frank Morley discovered a nice little theorem on the trisectors of a triangle, known as 'Morley's trisector theorem'. In the March issue Jan van de Craats and Jan Brinkhuis elucidated the role of cardioids in Morley's discovery of his theorem. In this article Jan van de Craats aims to answer the question *why* Morley was interested in studying cardioids at all.**

In 1899 the algebraic geometer Frank Morley (1860–1937) discovered a surprising result on the trisectors of a triangle. He mentioned it to friends, who spread it over the world in the form of mathematical gossip. Morley's trisector theorem, as it later became known, reads as follows (see Figure 1): *in any triangle, the three points of intersection of the adjacent angle trisectors form an equilateral triangle.*

In their recent paper 'Cardioids and Morley's trisector theorem' [1], Jan van de Craats and Jan Brinkhuis elucidated the role of cardioids in Morley's discovery of his theorem. Morley and his son Frank Vigor Morley also explained this connection on pages 239–244 of their book *Inversive Geometry* [4] from 1933, but this is no easy reading. The Morleys usually present their results in an informal way and rigorous proofs are seldom given. However, the book has been reprinted in 1954 and, again, in 2013, indicating that also the modern reader might find it valuable to study its contents.

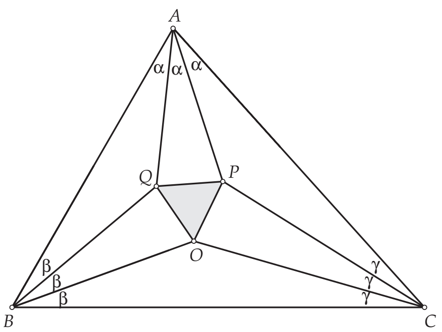


**Figure 1** Morley's trisector theorem.

In our paper [1], we took pains to explain Morley's reasoning on cardioids and trisectors in an accessible way and to provide detailed proofs. However, we didn't explain *why* Morley was interested in studying cardioids at all. The present paper, which in part is based on Morley's 1929 article [3] and on chapter XXI of *Inversive Geometry* [4], aims to answer this question. Its main results are summarized in Theorems 2, 3 and 4 and their proofs. In a strict sense, these theorems do not contain new results, but perhaps our presentation may inspire modern readers to excavate more jewels from Morley's work.

Morley considered a cardioid as a member of an infinite sequence of rational curves in the Argand plane (the Euclidean plane coordinated by complex numbers) $B_1, B_2, B_3, B_4, \dots$ in which $B_1$ is a point, $B_2$ a circle, $B_3$ a cardioid, and $B_4, B_5, \dots$ are 'higher' curves. The curves $B_n$ have many interesting properties, leading to intriguing theorems of a general nature with pleasant special cases. Morley's trisector theorem is just one instance. Another example is the five circles theorem illustrated in Figure 2. In Morley's words [4, p. 265]: *We place a ring of five circles with centers on a given circle and each intersecting the next on the circle. The five other intersections of the adjacent circles, being joined in order, form the five-line, and the salient thing is that the intersections of non-adjacent sides are also on the respective five circles.*

Later in this paper we will present a proof of this theorem as a special case of a more general result on curves of type $B_4$, but first we need to introduce some of Morley's idiosyncratic notations and terminology. The next two sections repeat in a condensed form a similar introduction in Van de Craats and Brinkhuis [1, pp. 26–28].

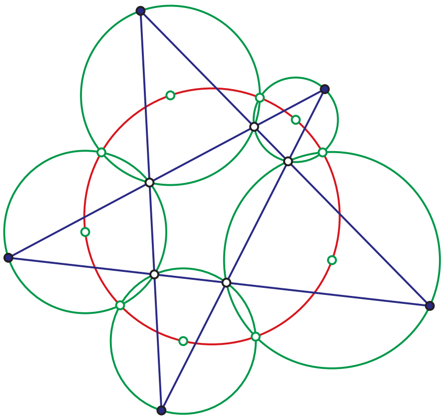
Figure 2 Morley's five circles theorem.

## Line-equations and map-equations

Points in the plane, viewed as complex numbers, will be represented by lower case letters. The letter $t$ will always be used for points on the unit circle, so $t\overline{t}=1$ in other words, $\overline{t}=1/t$. Such points will be called *turns*, since multiplication by $t$ amounts to an anticlockwise rotation around the origin over an angle $\arg(t)$. Occasionally, also the Greek letter $\tau$ will be used for turns.

For any two distinct points $x$ and $c$ on a line $L$ the vectors $x-c$ and $c-x$ both indicate the direction of $L$, but in opposite sense. However, the quotient $t=(x-c)/(\overline{x}-\overline{c})$, which is a point on the unit circle, is independent of the order of $x$ and $c$. In fact, it only depends on $L$ and not on the choice of $x$ and $c$ on $L$. It is a turn, called the *clinant* of $L$. Its argument equals *twice* the directed angle from the real axis to $L$. Two lines are parallel if and only if their clinants are equal. Furthermore, two lines are perpendicular if and only if their clinants differ by a factor $-1$.

If we fix the point $c$ on $L$ and consider $x$ as a variable, then the equation

$$x-c=t(\overline{x}-\overline{c}) \tag{1}$$

represents all points of $L$. Note that it is a *self-conjugate* equation: conjugation yields the same equation since $\overline{t}=1/t$.

If $L$ passes through the origin we may take $c=0$, leading to the simple equation $x=t\overline{x}$. If the origin is not on $L$, the image $b$ upon reflecting the origin in $L$ completely determines $L$. Since the line through the origin and $b$ is perpendicular to $L$, the clinant $t$ of $L$ equals $-b/\overline{b}$. Furthermore, the point $b/2$ is on $L$, so in equation (1) we may take $c=b/2$. The resulting equation

$$x-b/2=-(b/\overline{b})(\overline{x}-\overline{b}/2)$$

can be written as

$$\overline{b}x+b\overline{x}=b\overline{b}. \tag{2}$$

An equivalent way to express the points $x$ of the line $L$ is

$$x-b-t\overline{x}=0, \tag{3}$$

where $b=-t\overline{b}$. Note that equation (3) also holds if $L$ passes through the origin: then $b=0$ should be taken. Equations like (1), (2) and (3) will be called *line-equations*.

For any function $f(t)$ the equation $x=f(t)$ may be viewed as a parametric representation of a curve $\Gamma$ in the plane with parameter $t$ running through the unit circle. It will be called a *map-equation* of the curve $\Gamma$. In the sequel, $f(t)$ will always be a polynomial in $t$. In particular, the map-equation $x=c+at$ represents a circle with center $c$ and radius $|a|$.

## The line-equation of a cardioid

As explained in Van de Craats and Brinkhuis [1], for given $a\neq 0$ and $c$ the map-equation

$$x=c+2at-\overline{a}t^2 \tag{4}$$

describes a *cardioid* when the parameter $t$ runs through the unit circle (see Figure 3, where we have taken $c=0$ and $a=1$). Its name is derived from its heart-like shape. The point $c$, which is not on the curve, is called the *center* of the cardioid. Centers of cardioids played an important role in Morley's discovery of his trisector theorem. The cardioid has a *cusp* when $dx/dt=0$, which in Figure 3 occurs for parameter value $t=1$ at the point $x=1$, and in general for $t=a/\overline{a}$ at $x=c+a^2/\overline{a}$.

In [1] it is shown that the tangent line to the cardioid (4) at the point with parameter value $t$ is given by the line-equation

$$(x-c)-3at+3\overline{a}t^2-(\overline{x}-\overline{c})t^3=0. \tag{5}$$

Also this equation is self-conjugate, i.e., conjugation (and multiplying both sides by $-t^3$) yields the same equation.

For a given turn $t$, equation (5) represents the tangent line to the cardioid at the point with parameter value $t$. For varying $t$ we get the set of all tangent lines. The line-equation (5) thus yields an alternative representation of the cardioid (4), namely as the envelope of the set of its tangent lines.

The map-equation of the cardioid may be recovered from the line-equation by taking two 'neighboring' lines from this set, say, for parameter values $t$ and $\tau$, and subtracting their equations to get an equation for their point of intersection

$$\begin{aligned}&-3a(t-\tau)+3\overline{a}(t^2-\tau^2)\\&\quad-(\overline{x}-\overline{c})(t^3-\tau^3)=0.\end{aligned}$$

Dividing by $(t-\tau)$, taking the limit $\tau\to t$ and conjugating yields an equation for the point of tangency $x$,

$$-3\overline{a}+6a/t-3(x-c)/t^2=0,$$

from which it follows that

$$x=c+2at-\overline{a}t^2.$$

This, indeed, is the map-equation (4) of the cardioid. The method for obtaining the map-equation from the line-equation thus formally may be described as differentiation with respect to $t$, followed by conjugation and solving the resulting equation for $x$. In the sequel, we will always define curves by line-equations.

## The curves $B_n$

In general, for $n\geq 1$ we define a curve $B_n$ by the line-equation

$$\begin{aligned}B_n:\ &(x-c)-\binom{n}{1}a_1t+\binom{n}{2}a_2t^2-\cdots\\&+(-1)^{n-1}\binom{n}{n-1}a_{n-1}t^{n-1}\\&+(-1)^n(\overline{x}-\overline{c})t^n=0\end{aligned} \tag{6}$$

where $a_{n-k}=\overline{a_k}$ for all $k$ to ensure that the equation is self-conjugate. Note that for even $n$, say $n=2m$, this implies that the middle coefficient $a_m$ is real. The point $c$ is called the *center* of the curve. The reason for including the binomial coefficients will become clear when, below, we will introduce the so-called 'polarized' equation of $B_n$.

For a fixed turn $t$, equation (6) represents a line. Taking two distinct points $x_1$ and $x_2$ on this line, subtraction yields $(x_1-x_2)+(-1)^nt^n(\overline{x_1}-\overline{x_2})=0$, so the clinant of this line is $-(-1)^nt^n$. For vary-

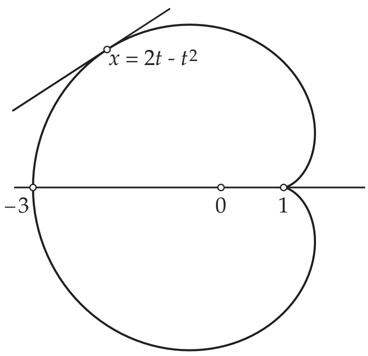

Figure 3 The cardioid $x=2t-t^2$ with a tangent line.

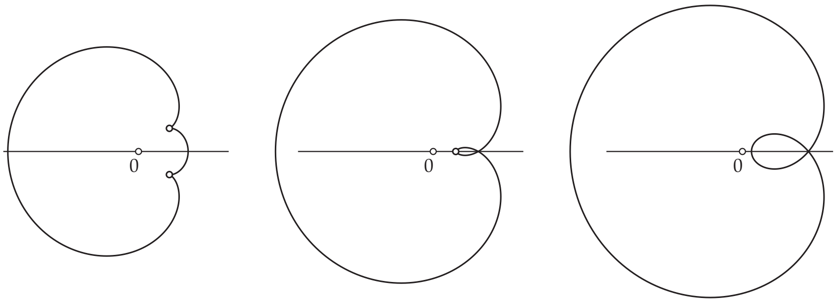


**Figure 4** Curves $B_4$ with two cusps (left), two coinciding cusps (middle) and no cusps (right).

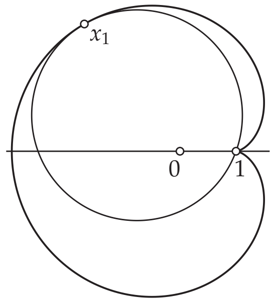


**Figure 5** Any osculant circle of a cardioid passes through its cusp.

ing $t$ we get a set of lines, the envelope of which is the curve $B_n$. The curve $B_n$ is completely determined by its center $c$ and the coefficients $a_k$.

As examples, we treat the cases $n = 2$, $1$ and $4$ in more detail.

For a curve $B_2$ we have the line-equation

$$B_2\colon\ (x-c)-2\rho t+(\overline{x}-\overline{c})\,t^2=0 \qquad (7)$$

where the coefficient $\rho$ should be real to keep the equation self-conjugate.

Differentiation with respect to $t$ yields $-2\rho+2(\overline{x}-\overline{c})\,t=0$, so by conjugation and solving for $x$ we get as the map-equation of the curve $x=c+\rho t$, which indeed is a circle. Its center is $c$ and its radius is $|\rho|$.For any fixed turn $t$, equation (7) represents the tangent line to the circle at the point with parameter value $t$. Its clinant is $-t^2$.

The line-equation of the 'curve' $B_1$ is

$$B_1\colon\ (x-c)-(\overline{x}-\overline{c})\,t=0.$$

For varying turns $t$ this is just the set of all lines through $c$, and the map-equation of $B_1$ is simply $x=c$. The clinant of a line from the line-equation is $t$.

For $n=4$ we get the line-equation

$$\begin{aligned}B_4\colon\ (x-c)-4at+6\mu t^2&\\ -4\overline{a}t^3+(\overline{x}-\overline{c})\,t^4&=0.\end{aligned}$$

Again, the middle coefficient $\mu$ must be real to keep the equation self-conjugate. Its clinant is $-t^4$. The map-equation now becomes

$$x=c+3at-3\mu t^2+\overline{a}t^3.$$

The cusp-equation $dx/dt=0$ is

$$a-2\mu t+\overline{a}t^2=0.$$

The reader might like to verify that the roots of this equation are turns if and only if $\mu^2-a\overline{a}\le 0$. In Figure 4 curves $B_4$ have been drawn with $c=0$, $a=1$ and $\mu=0.6$ (two cusps), $\mu=1$ (two coinciding cusps at $x=1$ for parameter value $t=1$) and $\mu=1.2$ (no cusps).

## Examples of osculant curves

To explain the concept of *osculant curves* of a curve $B_n$, we first take as an example a cardioid $B_3$. In its most simple form, its line-equation is given by

$$x-3t+3t^2-\overline{x}t^3=0.$$

Its map-equation is

$$x=2t-t^2,$$

so its center is $c=0$ and its cusp is $x=1$ (with parameter value $t=1$). By a change of coordinates, any non-degenerate cardioid can be written in this form.

For any three turns $t_1$, $t_2$, $t_3$, we associate to the line-equation of the cardioid the *polarized equation*

$$\begin{aligned}&x-(t_1+t_2+t_3)+(t_1t_2+t_2t_3+t_3t_1)\\ &-\overline{x}t_1t_2t_3=0.\end{aligned}$$

This again is a self-conjugate equation, so it represents a line associated with the three parameter values $t_1$, $t_2$ and $t_3$. If $t_1=t_2=t_3=t$, it is a tangent line to the cardioid, but if only $t_2=t_3=t$ we get the equation

$$x-(t_1+2t)+(t^2+2t_1t)-\overline{x}t_1t^2=0.$$

This, for fixed $t_1$ and varying $t$ may be viewed as the line-equation of a curve of type $B_2$. Indeed, differentiating with respect to $t$, conjugating and solving for $x$ yields

$$x=t_1+t-t_1t$$

which is the map-equation of the circle with center $t_1$ and radius $|1-t_1|$.

For $t=t_1$, the map-equations of both the circle and the cardioid yield the point $x_1=2t_1-t_1^2$, while also, in view of the line-equations, their tangent lines at this point coincide. Therefore, $x_1$ is a point where the two curves touch. The circle is called an *osculant circle* to the cardioid. Furthermore, the circle also passes through the cusp $x=1$ (take $t=1$). Thus we have the result that *any osculant circle of a cardioid passes through its cusp* (see Figure 5).

We now take two turns $t_1$ and $t_2$ and a variable turn $t$ in the polarized equation

$$\begin{aligned}&x-(t_1+t_2+t)+(t_1t_2+t_2t+t_1t)\\ &-\overline{x}t_1t_2t=0.\end{aligned} \qquad (8)$$

This is the line-equation of a 'curve' of type $B_1$, which, as a map-equation, is just the point

$$x_{12}=t_1+t_2-t_1t_2.$$

It is the second intersection point of the osculant circles for $t_1$ and $t_2$ (their other intersection point is the cusp $x=1$). The point $x_{12}$ is an *osculant* of both the osculant circles; it is called a *second osculant* of the cardioid (the first osculants being the osculant circles).

For $t=t_3$ the line-equation (8) of the second osculant yields the fully polarized form

$$\begin{aligned}&x-(t_1+t_2+t_3)+(t_1t_2+t_2t_3+t_3t_1)\\ &-\overline{x}t_1t_2t_3=0.\end{aligned}$$

This is a line through $x_{12}$, which, by symmetry, also passes through $x_{23}$ and $x_{31}$. It is called a *third osculant* of the cardioid.

We thus have proved: *for any three osculant circles of a cardioid, their second intersection points are collinear* (as we have seen, all osculant circles also pass through the cusp). See Figure 6.

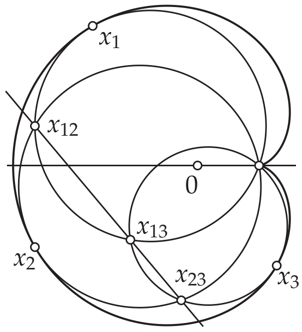


**Figure 6** Three osculant circles of a cardioid. They all pass through the cusp, and their second intersections are collinear.

As a second example, consider the osculants of a circle given by the line-equation

$$(x-c)-2\rho t+(\overline{x}-\overline{c})t^2=0 \quad \text{(with real } \rho)$$

(cf. (7)). Recall that its map-equation is $x=c+\rho t$. For any two turns $t_1$ and $t_2$ the polarized equation yields the osculant line

$$L_{12}:\ (x-c)-\rho(t_1+t_2)+(\overline{x}-\overline{c})t_1t_2=0.$$

For fixed $t_1$ and variable $t$ we get the line-equation of a first osculant, which is a 'curve' of type $B_1$:

$$(x-c)-\rho(t_1+t)+(\overline{x}-\overline{c})t_1t=0.$$

As a map-equation, this is just the point $x_1=c+\rho t_1$ on the circle. Similarly, for variable $t$ and fixed $t_2$, we get the line-equation

$$(x-c)-\rho(t+t_2)+(\overline{x}-\overline{c})tt_2=0$$

which, as a map-equation, is just the point $x_2=c+\rho t_2$ on the circle. The osculant line $L_{12}$, therefore, is the line through $x_1$ and $x_2$.

**Osculant curves in general**

Let $n\geq 2$. Take a general curve $B_n$ with line-equation (6), or, written in a more compact form,

$$x-c+\sum_{k=1}^{n-1}(-1)^k\binom{n}{k}a_kt^k+(-1)^n(\overline{x}-\overline{c})t^n=0 \qquad (9)$$

where $a_{n-k}=\overline{a_k}$ for all $k$.

For $n$ turns $t_1,\ldots,t_n$, the polarized form of the line-equation is

$$x-c+\sum_{k=1}^{n-1}(-1)^ka_k\sigma_k+(-1)^n(\overline{x}-\overline{c})\sigma_n=0 \qquad (10)$$

where the $\sigma_k$ are the symmetric functions of $t_1,\ldots,t_n$ defined by

$$\begin{aligned}\sigma_1&=t_1+t_2+\cdots+t_n\\ \sigma_2&=t_1t_2+\cdots+t_{n-1}t_n\\ \sigma_3&=t_1t_2t_3+\cdots+t_{n-2}t_{n-1}t_n\\ &\vdots\\ \sigma_n&=t_1t_2\cdots t_n.\end{aligned}$$

It will be clear how in general the osculant curves of $B_n$ will be defined: fix $m$ of the parameter values $t_i$ in the polarized equation and take the others equal to $t$. Then a curve of type $B_{n-m}$ is obtained, an $m$-th osculant of $B_n$. The $n$-th osculant is the line given by the fully polarized line-equation (10).

To obtain the map-equation of $B_n$, we differentiate (9) with respect to $t$,

$$\sum_{k=1}^{n-1}(-1)^kk\binom{n}{k}a_kt^{k-1}+(-1)^nn(\overline{x}-\overline{c})t^{n-1}=0.$$

By using $k\binom{n}{k}=n\binom{n-1}{k-1}$, division by $n$, conjugation and multiplication by $t^{n-1}$ we get

$$\sum_{k=1}^{n-1}(-1)^k\binom{n-1}{k-1}\overline{a_k}t^{n-k}+(-1)^n(x-c)=0.$$

Since $\overline{a_k}=a_{n-k}$, the map-equation of $B_n$ can be written as

$$x=c-\sum_{k=1}^{n-1}(-1)^{n-k}\binom{n-1}{k-1}a_{n-k}t^{n-k}.$$

Using $\binom{n-1}{k-1}=\binom{n-1}{n-k}$ and writing $k$ instead of $n-k$ we get

$$x=c-\sum_{k=1}^{n-1}(-1)^k\binom{n-1}{k}a_kt^k. \qquad (11)$$

Its derivative with respect to $t$ is

$$\begin{aligned}\frac{dx}{dt}&=-\sum_{k=1}^{n-1}(-1)^kk\binom{n-1}{k}a_kt^{k-1}\\ &=-(n-1)\sum_{k=1}^{n-1}(-1)^k\binom{n-2}{k-1}a_kt^{k-1}\end{aligned} \qquad (12)$$

so the cusps of $B_n$ (if any) satisfy the *cusp-equation*

$$\sum_{k=1}^{n-1}(-1)^k\binom{n-2}{k-1}a_kt^{k-1}=0. \qquad (13)$$

For a parameter value $t_1$ the first osculant curve is given by the line-equation that results from taking $t_2=\cdots=t_n=t$ in the polarized equation (10):

$$x-c+\sum_{k=1}^{n-1}(-1)^k\left\{\binom{n-1}{k}t^k+\binom{n-1}{k-1}t_1t^{k-1}\right\}a_k+(-1)^n(\overline{x}-\overline{c})t_1t^{n-1}=0. \qquad (14)$$

Its map-equation, obtained in the usual way by differentiation, conjugation, solving for $x$ and writing $k$ instead of $n-k$, is

$$x=c-\sum_{k=1}^{n-1}(-1)^k\left\{\binom{n-2}{k-1}t_1t^{k-1}+\binom{n-2}{k}t^k\right\}a_k. \qquad (15)$$

This is a curve of type $B_{n-1}$ with center $x=c+a_1t_1$, so all centers of the first osculants are on the *centric circle*, the circle with map-equation

$$x=c+a_1t.$$

For $t=t_1$, both the line-equations (9) of the curve $B_n$ and (14) of the first osculant curve yield the same line, while also the map-equations (11) and (15) yield the same point, so at this point the two curves touch. But, as we have seen already in the case of the cardioid, *the first osculant also passes through the cusps (if any) of $B_n$*. To prove this, we write the map-equation (11) of $B_n$ in the form

$$x=c-\sum_{k=1}^{n-1}(-1)^k\left\{\binom{n-2}{k-1}+\binom{n-2}{k}\right\}a_kt^k.$$

For any $t_c$ satisfying the cusp-equation (13), both the map-equations (11) of $B_n$ and (15) of the osculant curve yield

$$x=c-\sum_{k=1}^{n-1}(-1)^k\binom{n-2}{k}a_kt_c^k$$

as desired.

The higher osculants yield more complicated formulas, but at this point we can collect a general result, starting from the bottom end, the osculant lines given by fully polarized equations like (10). Let a curve $B_n$ and $n+1$ parameter values $t_1,\ldots,t_{n+1}$ be given. Any $n$ of these determine an osculant line, $n+1$ lines in total. Any $n-1$ parameter values determine an osculant point, the intersection of two osculant lines. Any $n-2$ parameter values determine an osculant circle, the circumscribed circle of three osculant points, which is also the circumscribed circle of the triangle formed by the corresponding three osculant lines. Any $n-3$ parameter values determine an osculant cardioid, with four osculant circles through its cusp. And so on.

Thus, a curve $B_n$ and $n+1$ parameter values determine $n+1$ lines with lots of interesting properties. In a later section we will show that the situation with respect to the lines is completely general, since we will prove that *any $n+1$ lines, no two parallel, determine a unique curve $B_n$ for which they are osculant lines.* But first we will study the osculants of curves $B_4$ in more detail, since these contain interesting special cases.

**Osculants of a curve $B_4$**

Let a curve $B_4$ be given. Without loss of generality, we may assume that its center is $c=0$. Then its line-equation is

$$B_4:\ x-4at+6\mu t^2-4\overline{a}t^3+\overline{x}t^4=0 \qquad (16)$$

with real $\mu$. Its map-equation is

$$x=3at-3\mu t^2+\overline{a}t^3. \qquad (17)$$

The cusp-equation $dx/dt=0$ yields

$$a-2\mu t+\overline{a}t^2=0. \qquad (18)$$

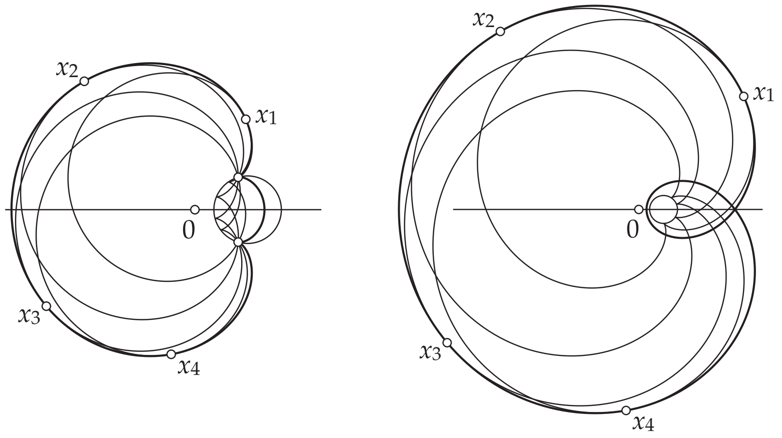


**Figure 7** Curves $B_4$ with and without cusps, together with four osculant cardioids and the cuspoidal circle.

If the discriminant $\mu^2 - a\overline{a}$ is negative or zero, the solutions of the cusp-equation are turns and $B_4$ has two (possibly coinciding) cusps.

For any four parameter values $t_1, t_2, t_3, t_4$, the polarized form

$$x - a(t_1+t_2+t_3+t_4) + \mu(t_1 t_2 + \cdots) \\ - \overline{a}(t_1 t_2 t_3 + \cdots) + \overline{x} t_1 t_2 t_3 t_4 = 0$$

represents an osculant line $L_{1234}$. It contains the four osculant points $x_{123}$, $x_{124}$, $x_{134}$, $x_{234}$, where, e.g.,

$$x_{123} = a(t_1+t_2+t_3) - \mu(t_1 t_2 + t_2 t_3 + t_3 t_1) \\ + \overline{a} t_1 t_2 t_3 .$$

Any two parameter values determine an osculant circle like $K_{12}$, given by the map-equation

$$K_{12}: \quad x = a(t_1+t_2+t) - \mu t_1 t_2 \\ - \mu(t_1+t_2)t + \overline{a} t_1 t_2 t.$$

Its center is $m_{12} = a(t_1+t_2) - \mu t_1 t_2$ and its radius is $|a - \mu(t_1+t_2) + \overline{a} t_1 t_2|$. The circle $K_{12}$ contains the osculant points $x_{123}$ and $x_{124}$ and passes through the cusps of the osculant cardioids $C_1$ and $C_2$, where, e.g., $C_1$ is given by the map-equation

$$C_1: \quad x = at_1 + 2(a - \mu t_1)t + (\overline{a} t_1 - \mu)t^2. \quad (19)$$

As we have proved above, the cardioid $C_1$ passes through each of the cusps of $B_4$ (if any).

The center of $C_1$ is $x = at_1$ and the centric circle (the circle containing all centers of osculant cardioids) thus has map-equation $x = at$.

The parameter value $t_c$ of the cusp of $C_1$ is the root of its cusp-equation

$$a - \mu(t_1 + t) + \overline{a} t_1 t = 0. \quad (20)$$

We will show now that if $\mu \neq 0$, *the cusps of the osculant cardioids, together with the cusps of $B_4$ (if any), are on a circle, the so-called cuspoidal circle, given by the map-equation*

$$x = (1/\mu)(a^2 + (a\overline{a} - \mu^2)t). \quad (21)$$

To prove this, note that for $\mu \neq 0$, the map-equation (19) of $C_1$ can be written as

$$\mu x = (\mu t - a)(a - \mu(t_1+t) + \overline{a} t_1 t) \\ + a^2 + (a\overline{a} - \mu^2) t_1 t$$

so if $t_c$ is the root of the cusp-equation (20) of $C_1$, then

$$\mu x = a^2 + (a\overline{a} - \mu^2) t_1 t_c$$

which, indeed, defines a point on the cuspoidal circle (21).

It might happen that $t_1$ is a root of the cusp equation (18) of $B_4$. Then, manifestly, $t_1$ is also the root of the cusp-equation (20) of the osculant cardioid $C_1$. The cusps of $B_4$ (if any) therefore also occur as cusps of osculant cardioids, so the cuspoidal circle also passes through the cusps of $B_4$. See Figure 7, where curves $B_4$ are shown with $a = 1$ and, respectively, $\mu = 0.6$ (two cusps) and $\mu = 1.25$ (no cusps). In each case, four osculant cardioids have been drawn together with the cuspoidal circle.

### Morley's five circles theorem

It might happen that in the line-equation (16) of the curve $B_4$ we have $a = 0$, so that the map-equation of $B_4$ reduces to

$$x = -3\mu t^2$$

where $\mu$ is a real number. This is a circle described twice. If $\mu = 0$, then $B_4$ degenerates into a point. Leaving this case aside, we may suppose without loss of generality that $\mu = 1$ in which case $B_4$ is a circle with radius $3$ described twice.

The first osculant $C_1$ then is the cardioid with map-equation

$$x = -2t_1 t - t^2$$

(cf. equation (19)). Its center is the origin, its cusp is $c_1 = t_1^2$, so the cuspoidal circle is the unit circle. In Figure 8 we see $B_4$ with five osculant cardioids at points $x_1, \ldots, x_5$, respectively, and the cuspoidal circle.

The second osculant $K_{12}$ is the circle with map-equation

$$x = -(t_1+t_2)t - t_1 t_2 .$$

It has center $-t_1 t_2$ and radius $|t_1 + t_2|$. The center thus is on the unit circle, which is also the cuspoidal circle. Furthermore, for $t = -t_1$ we get the cusp $t_1^2$ of $C_1$ while $t = -t_2$ yields the cusp $t_2^2$ of $C_2$, so, indeed, the circle $K_{12}$ passes through the cusps of $C_1$ and $C_2$.

The third osculant $x_{123}$ is the point

$$x_{123} = -(t_1 t_2 + t_2 t_3 + t_3 t_1).$$

It is the common point of the osculant circles $K_{12}$, $K_{23}$ and $K_{31}$.

For five parameter values $t_1, \ldots, t_5$ we have five osculant cardioids $C_1, \ldots, C_5$, ten osculant circles $K_{12}, \ldots$, each tangent to two osculant cardioids, ten osculant points $x_{123}, \ldots$, each on three osculant circles and, finally, five osculant lines $L_{1234}, L_{1235}, L_{1245}, L_{1345}, L_{2345}$, each containing four osculant points. For example, $L_{1234}$ contains the points $x_{123}$, $x_{124}$, $x_{134}$ and $x_{234}$. Figure 9 shows the curve $B_4$ with $a = 0$, $\mu = 1$, the cuspoidal circle in red, the five osculant cardioids with their cusps in yellow, the ten osculant circles with their centers on the cuspoidal circle in green, the ten osculant points and the five osculant lines in blue. The reader is invited to identify the osculant cardioids with their cusps, the osculant circles with their cen-

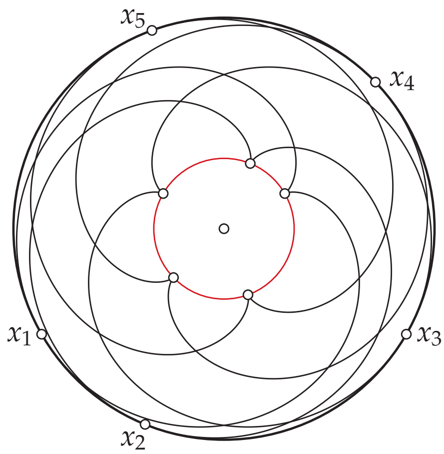


**Figure 8** The curve $B_4$ with $a = 0$ and $\mu = 1$ with five osculant cardioids and the cuspoidal circle.

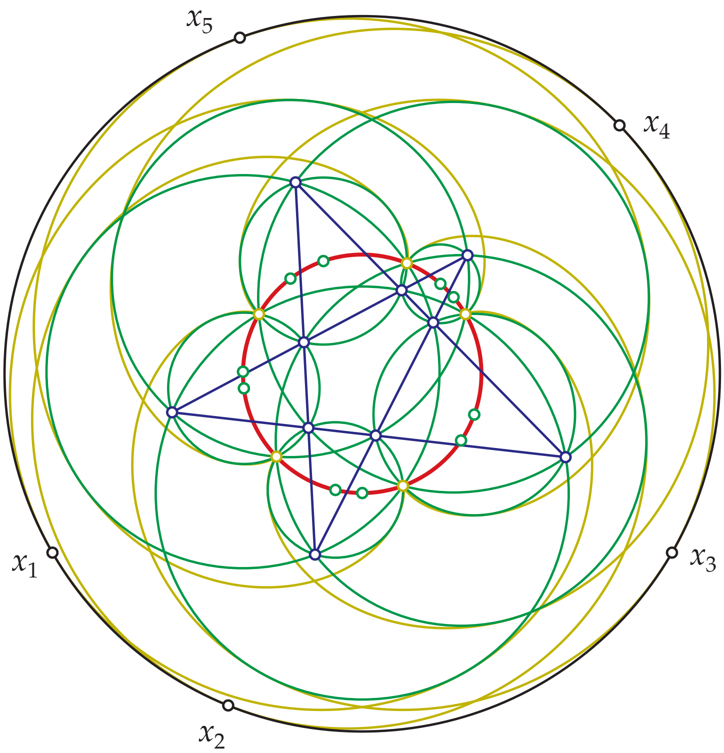


**Figure 9** The curve $B_4$ with $a = 0$, $\mu = 1$ (black) with the cuspoidal circle (red). Furthermore, for five parameter values we have drawn the five osculant cardioids with their cusps (yellow), the ten osculant circles with their centers (green), the ten osculant points and the five osculant lines (blue).

ters, the osculant points and the osculant lines.

In Figure 9, the five osculant lines form a pentagon with four osculant points on each (extended) side. For determining the pentagon, it is sufficient to draw the cuspoidal circle and only five of the ten osculant circles, as shown in Figure 10. This yields Morley's five circles theorem, as announced in the introduction and illustrated in Figure 2.

**Theorem 1** (Morley's five circles theorem). *If five circles are chosen with their centers on a given circle such that each intersects the next on the circle, then the five lines through the other intersection points of adjacent circles intersect again on the respective circles.*

*Proof.* Let the given circle be the unit circle and let $K_{12}$, $K_{23}$, $K_{34}$, $K_{45}$, $K_{51}$ be the five circles with centers $m_{i,i+1}$ on the unit circle, each circle $K_{i-1,i}$ intersecting the next one $K_{i,i+1}$ on the unit circle in point $c_i$ (indices modulo 5, see Figure 10). Note that $m_{i,i+1}/c_i = c_{i+1}/m_{i,i+1}$, so $m_{i,i+1}^2 = c_i c_{i+1}$.

We will show now that it is possible to choose turns $t_1$, $t_2$, $t_3$, $t_4$, $t_5$ such that the five circles $K_{i,i+1}$ are the second osculant curves of the curve

$$B_4\colon\ x + 6t^2 + \overline{x}t^4 = 0$$

for consecutive parameter pairs $(t_i, t_{i+1})$. Then the centers of these circles should be $m_{12} = -t_1 t_2$, $m_{23} = -t_2 t_3$, $m_{34} = -t_3 t_4$, $m_{45} = -t_4 t_5$, $m_{51} = -t_5 t_1$, respectively, while $c_1 = t_1^2$, $c_2 = t_2^2$, $c_3 = t_3^2$, $c_4 = t_4^2$, $c_5 = t_5^2$, the point $c_i$ being the cusp of the osculant cardioid $C_i$ of $B_4$ for parameter value $t_i$.

To determine the turns $t_i$ from the centers $m_{i,i+1}$ and the intersection points $c_i$, we choose $t_1$ as one of the two square roots of $c_1$ and define $t_2 = -m_{12}/t_1$, $t_3 = -m_{23}/t_2$, $t_4 = -m_{34}/t_3$, $t_5 = -m_{45}/t_4$. It then follows from $c_1 = t_1^2$ and $m_{i,i+1}^2 = c_i c_{i+1}$ that also $c_i = t_i^2$ for $i = 2, 3, 4, 5$, as desired.

With these parameter values $t_i$ it is possible to construct all osculant cardioids, circles, points and lines of $B_4$ as in Figure 9. Among these osculants, we find the five osculant circles, the ten osculant points and the five osculant lines of Figure 10, with three osculant points on each circle and four osculant points on each osculant line, as indicated above. For instance, circle $K_{12}$ contains the osculant points $x_{123}$, $x_{124}$ and $x_{125}$, while line $L_{1234}$ contains $x_{123}$, $x_{124}$, $x_{134}$ and $x_{234}$. This establishes Morley's five circles theorem. □

**A curve $B_4$ with a cuspoidal segment**

Take a curve $B_4$ given by (16) with $a = 1$ and $\mu = 0$. Then its map-equation is

$$x = 3t + t^3.$$

The curve $B_4$ has two cusps, at $x = \pm 2i$, taken for $t = \pm i$. We will prove that in this case the cuspoidal circle degenerates into the segment connecting the cusps $\pm 2i$ of $B_4$.

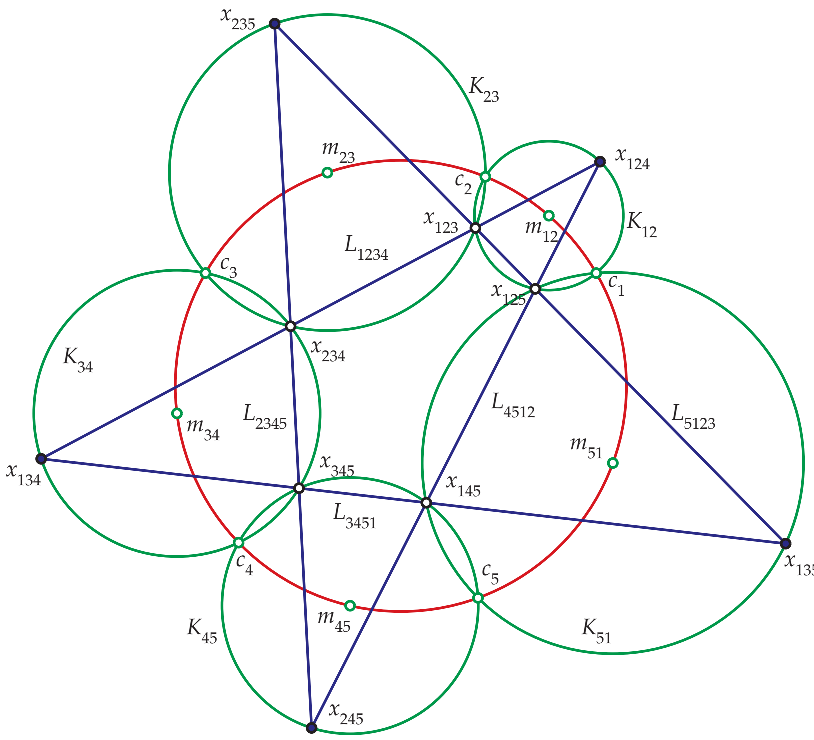


**Figure 10** Morley's five circles theorem in relation to Figure 9.

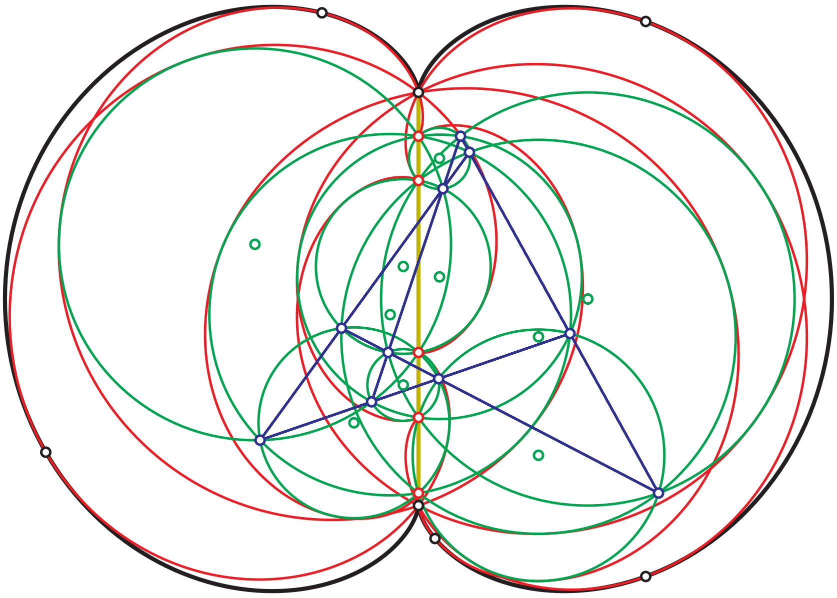

**Figure 11** The curve $B_4$ with $a=1$, $\mu=0$, the cuspoidal segment (yellow), five osculant cardioids with their cusps (red), the corresponding ten osculant circles with their centers (green), the ten osculant points and the five osculant lines (blue).

Any osculant cardioid $C_1$ with map-equation

$$x = t_1 + 2t + t_1 t^2$$

passes for $t=\pm i$ through the cusps $\pm 2i$ of $B_4$. Furthermore, the cusp of $C_1$ is obtained from $dx/dt=0$, which yields $t=-1/t_1$, so the cusp of $C_1$ is $t_1 - 1/t_1$. This, indeed, is a point on the segment $[-2i, 2i]$. Therefore, the cuspoidal segment connects the cusps of $B_4$ (see Figure 11).

The osculant circle $K_{12}$ with map-equation

$$x = t_1 + t_2 + t + t_1 t_2 t$$

has center $m_{12} = t_1 + t_2$ and radius $|1 + t_1 t_2|$. It contains the cusps of the cardioids $C_1$ and $C_2$.

The osculant point $x_{123}$, given by

$$x_{123} = t_1 + t_2 + t_3 + t_1 t_2 t_3,$$

is on the osculant circles $K_{12}$, $K_{23}$ and $K_{31}$.

The four osculant points $x_{123}$, $x_{124}$, $x_{134}$, $x_{234}$ are on the osculant line $L_{1234}$, given by the line-equation

$$\begin{aligned} &x - (t_1 + t_2 + t_3 + t_4) \\ &\quad - (t_1 t_2 t_3 + t_1 t_2 t_4 + t_1 t_3 t_4 + t_2 t_3 t_4) \\ &\quad + \overline{x} t_1 t_2 t_3 t_4 = 0 \end{aligned}$$

which is a fully polarized form of the line-equation of $B_4$.

In Figure 11 we have drawn the curve $B_4$ (black) and the cuspoidal segment (yellow) and, for five parameter values $t_1$, $t_2$, $t_3$, $t_4$, $t_5$, the osculant cardioids $C_1, \ldots$ with their cusps (red), the ten osculant circles $K_{12}, \ldots$ with their centers (green), the ten osculant points $x_{123}, \ldots$ and the five osculant lines $L_{1234}, \ldots$ (blue).

### Curves $B_n$ determined by $n+1$ lines

**Theorem 2.** *Suppose that $n+1$ lines, no two parallel, are given. Then there exists a unique curve $B_n$ for which the given lines are osculant lines.*

*Proof.* For clearness we will give the proof for $n=3$, but in such a way that it is obvious that the general proof for $n \geq 2$ proceeds in a similar way. Note that the case $n=1$ is trivial: the 'curve' $B_1$ then is the intersection point of the two lines.

Therefore, let four arbitrary lines $L_1, L_2, L_3, L_4$ be given, no two parallel, and let their clinants be the (distinct) turns $\tau_1$, $\tau_2, \tau_3$, $\tau_4$. Without loss of generality, we may assume that $\tau_1\tau_2\tau_3\tau_4 = 1$. Let $b_i$ be the image of the origin upon reflection in line $L_i$. Then line $L_i$ is given by the self-conjugate equation $x - b_i - \tau_i \overline{x} = 0$, where $b_i = -\tau_i \overline{b_i}$.

For ease of computation, we will work with the turns $t_i = \overline{\tau_i} = \tau_i^{-1}$. Then, in view of $\tau_1\tau_2\tau_3\tau_4 = 1$, the clinants can also be written as $t_2t_3t_4$, $t_1t_3t_4$, $t_1t_2t_4$, $t_1t_2t_3$ and the four lines become

$$\begin{aligned} x - b_1 - \overline{x} t_2 t_3 t_4 &= 0, \\ x - b_2 - \overline{x} t_1 t_3 t_4 &= 0, \\ x - b_3 - \overline{x} t_1 t_2 t_4 &= 0, \\ x - b_4 - \overline{x} t_1 t_2 t_3 &= 0, \end{aligned}$$

where

$$\overline{b_i} = -b_i t_i.$$

The general line-equation of a cardioid is

$$x - c - 3at + 3\overline{a}t^2 - (\overline{x} - \overline{c})t^3 = 0$$

and if the four given lines are the osculant lines for parameter values $t_1$, $t_2$, $t_3$, $t_4$ of this cardioid, then their line-equations can also be written as

$$\begin{aligned} &x - c - a(t_2 + t_3 + t_4) + \overline{a}(t_3t_4 + t_4t_2 + t_2t_3) \\ &\quad - (\overline{x} - \overline{c})\, t_2t_3t_4 = 0, \\ &x - c - a(t_1 + t_3 + t_4) + \overline{a}(t_3t_4 + t_4t_1 + t_1t_3) \\ &\quad - (\overline{x} - \overline{c})\, t_1t_3t_4 = 0, \\ &x - c - a(t_1 + t_2 + t_4) + \overline{a}(t_2t_4 + t_4t_1 + t_1t_2) \\ &\quad - (\overline{x} - \overline{c})\, t_1t_2t_4 = 0, \\ &x - c - a(t_1 + t_2 + t_3) + \overline{a}(t_2t_3 + t_3t_1 + t_1t_2) \\ &\quad - (\overline{x} - \overline{c})\, t_1t_2t_3 = 0. \end{aligned}$$

Therefore, $b_1$, $b_2$, $b_3$ and $b_4$ should satisfy

$$\begin{aligned} b_1 &= c + a(t_2 + t_3 + t_4) - \overline{a}(t_3t_4 + t_4t_2 + t_2t_3) \\ &\quad - \overline{c}\, t_2t_3t_4, \text{ et cetera.} \end{aligned}$$

In other words, the 4-tuple $(c, a, -\overline{a}, -\overline{c})$ should be a solution of the following system of four equations in the unknown $z_1$, $z_2$, $z_3$, $z_4$,

$$\begin{aligned} b_1 &= z_1 + z_2(t_2 + t_3 + t_4) \\ &\quad + z_3(t_3t_4 + t_4t_2 + t_2t_3) + z_4 t_2t_3t_4, \\ b_2 &= z_1 + z_2(t_1 + t_3 + t_4) \\ &\quad + z_3(t_3t_4 + t_4t_1 + t_1t_3) + z_4 t_1t_3t_4, \\ b_3 &= z_1 + z_2(t_1 + t_2 + t_4) \\ &\quad + z_3(t_2t_4 + t_4t_1 + t_1t_2) + z_4 t_1t_2t_4, \\ b_4 &= z_1 + z_2(t_1 + t_2 + t_3) \\ &\quad + z_3(t_2t_3 + t_3t_1 + t_1t_2) + z_4 t_1t_2t_3. \end{aligned}$$

Conjugating this system, using $t_1t_2t_3t_4 = 1$ and $\overline{b_i} = -b_i t_i$, we get the equivalent system

$$\begin{aligned} -b_1 &= \overline{z_1} t_2t_3t_4 + \overline{z_2}(t_3t_4 + t_4t_2 + t_2t_3) \\ &\quad + \overline{z_3}(t_2 + t_3 + t_4) + \overline{z_4}, \\ -b_2 &= \overline{z_1} t_1t_3t_4 + \overline{z_2}(t_3t_4 + t_4t_1 + t_1t_3) \\ &\quad + \overline{z_3}(t_1 + t_3 + t_4) + \overline{z_4}, \\ -b_3 &= \overline{z_1} t_1t_2t_4 + \overline{z_2}(t_2t_4 + t_4t_1 + t_1t_2) \\ &\quad + \overline{z_3}(t_1 + t_2 + t_4) + \overline{z_4}, \\ -b_4 &= \overline{z_1} t_1t_2t_3 + \overline{z_2}(t_2t_3 + t_3t_1 + t_1t_2) \\ &\quad + \overline{z_3}(t_1 + t_2 + t_3) + \overline{z_4}, \end{aligned}$$

showing that with any solution $(z_1, z_2, z_3, z_4)$ also $(-\overline{z_4}, -\overline{z_3}, -\overline{z_2}, -\overline{z_1})$ is a solution. In particular, when the system has *only one* solution, then these two solutions must be identical, in other words, then the solution must be of the form $(c, a, -\overline{a}, -\overline{c})$ for certain $a$ and $c$, as desired. Therefore, we only have to show that the determinant of this system is non-zero. Using the symmetric functions

$$\sigma_1 = t_1+t_2+t_3+t_4,$$
$$\sigma_2 = t_1t_2+t_1t_3+t_1t_4+t_2t_3+t_2t_4+t_3t_4,$$
$$\sigma_3 = t_1t_2t_3+t_1t_2t_4+t_1t_3t_4+t_2t_3t_4,$$

the determinant can be written as

$$|1\ \ (\sigma_1 - t_i)\ \ (\sigma_2 - t_i(\sigma_1 - t_i))\ \ (\sigma_3 - t_i(\sigma_2 - t_i(\sigma_1 - t_i)))|.$$

But it is easy to see that this determinant is equal to the Vandermonde determinant

$$|1\ t_i\ t_i^2\ t_i^3| = \prod_{k>j}(t_k - t_j)$$

which indeed is non-zero if all $t_i$ are distinct.

This completes the proof that any four lines, no two parallel, determine a unique cardioid for which they are osculant lines. But in a similar way a proof can be given for any $n \geq 2$, the only difference being an extra minus sign in the determinant for even $n$. □

**Curves $B_n$ tangent to $n+1$ lines**

Given three lines forming the extended sides of a triangle, there are four circles for which these lines are tangent lines. More generally, we have the following theorem:

**Theorem 3.** *Let $n \geq 2$ and suppose that $n+1$ lines, no two parallel, are given. Then there are at least one and at most $n^n$ curves of type $B_n$ tangent to each of the given lines.*

*Proof.* Let the given lines be $L_0, \ldots, L_n$ and let the distinct turns $\tau_0, \ldots, \tau_n$ be their clinants. If $b_i$ is the image of the origin upon reflection in line $L_i$, then $b_i = -\tau_i \overline{b_i}$ and the line-equation of line $L_i$ is

$$x - b_i - \tau_i \overline{x} = 0. \qquad (22)$$

If these lines are tangent lines to a curve $B_n$ there must be constants $c, a_1, \ldots, a_{n-1}$ with $\overline{a_k} = a_{n-k}$ for all $k$, and parameter values $t_0, \ldots, t_n$, such that line $L_i$ is given by the line-equation

$$L_i\colon\ \ x - c + \sum_{k=1}^{n-1}(-1)^k\binom{n}{k}a_k t_i^k + (\overline{x} - \overline{c})(-1)^n t_i^n = 0$$

(cf. the line-equation (9) of a curve of type $B_n$). This should be the same as (22), so, in the first place, $-\tau_i = (-t_i)^n$ must hold, which leaves us with $n$ choices for each $t_i$. Once we have chosen such parameter values $t_i$, the constants $c$ and $a_k$ should also satisfy the system of $n+1$ equations

$$b_i = c - \sum_{k=1}^{n-1}(-1)^k\binom{n}{k}a_k t_i^k + \overline{c}(-1)^n t_i^n$$

with $a_{n-k} = \overline{a_k}$ for all $k$. In other words, the $(n+1)$-tuple $(c, a_1, a_2, \ldots, \overline{a_2}, \overline{a_1}, \overline{c})$ should be a solution of the system of $n+1$ equations in the unknown $z_0, \ldots, z_n$,

$$b_i = z_0 - \sum_{k=1}^{n-1}(-1)^k\binom{n}{k}t_i^k z_k + (-1)^n t_i^n z_n \quad (i = 0, \ldots, n). \qquad (23)$$

Up to a non-zero constant, the determinant of this system is the Vandermonde determinant

$$|1\ t_i\ t_i^2\ \cdots\ t_i^n| = \prod_{k>j}(t_k - t_j)$$

which is non-zero since we have assumed that all clinants $\tau_i$, and therefore also all parameter values $t_i$, are distinct. It follows that the system (23) has a unique solution $(z_0, \ldots, z_n)$.

Conjugating the system (23), using $\overline{b_i} = -b_i/\tau_i = b_i/(-t_i)^n$, $\binom{n}{k} = \binom{n}{n-k}$ and multiplying equation $i$ by $(-t_i)^n$ yields the equivalent system

$$b_i = (-t_i)^n \overline{z_0} - \left(\sum_{k=1}^{n-1}\binom{n}{n-k}(-1)^{n-k}t_i^{n-k}\overline{z_k}\right) + \overline{z_n}$$

which, with the notation $n - k = j$, can be rewritten as

$$b_i = \overline{z_n} - \left(\sum_{j=1}^{n-1}(-1)^j\binom{n}{j}t_i^j\overline{z_{n-j}}\right) + (-1)^n t_i^n \overline{z_0}$$

showing that with each solution $(z_0, \ldots, z_n)$ of the system (23) also $(\overline{z_n}, \ldots, \overline{z_0})$ is a solution. But since the solution of system (23) is unique, it must be of the form $(c, a_1, a_2, \ldots, \overline{a_2}, \overline{a_1}, \overline{c})$, as desired.

In choosing the parameter values $t_i$ satisfying $-\tau_i = (-t_i)^n$, we have $n$ choices for each $t_i$. However, since multiplying each $t_i$ by the same number $\omega^k$, where $\omega = e^{2\pi i/n}$, yields *the same* curve $B_n$, there are not $n^{n+1}$, but at most $n^n$ curves $B_n$ tangent to $n+1$ given lines, no two parallel. Note that the actual number may be smaller than $n^n$, for example if the $n+1$ lines all pass through a common point, say the origin, so $b_i = 0$ for all $i$. Then $c = 0$ and $a_k = 0$ for all $k$ and the only curve of type $B_n$ that is tangent to all $n+1$ lines is the degenerated 'curve' given by the line-equation

$$x + \overline{x}(-1)^n t^n = 0$$

which, as a map-equation, is just the origin itself. Since the system (23) is non-singular, there is always at least one curve of type $B_n$ tangent to all $n+1$ lines. □

**The axes of a system of $n$ lines**

In the former section we have seen that there are at least one and at most $n^n$ curves of type $B_n$ tangent to each of $n+1$ given lines, no two parallel. However, if we remove one of these lines, there will be an infinitude of inscribed $n$-curves, i.e., curves of type $B_n$ tangent to each of the $n$ remaining lines.

**Theorem 4.** *Let $n \geq 2$ and suppose that $n$ lines are given, no two parallel. Then the locus of the centers of the inscribed n-curves consists of a set of at most $n^{n-1}$ lines, occurring in n equispaced directions, with the same number of lines in each direction.*

Morley [3, pp. 468–469] and [4, chapter XXI] called these lines the *axes* of the given system of lines. For the main ideas in the following proof, I am indebted to my colleague Henk Pijls of the University of Amsterdam.

*Proof.* Let $L_1, \ldots, L_n$ be the given lines. Add one more line $L_0$ to the system, not parallel to any of the given lines. Let $\tau_i$ be the clinant of $L_i$ and let $b_i$ be the image of the origin upon reflection in line $L_i$ $(i = 0, 1, \ldots, n)$. For each $i$, let a turn $t_i$ be chosen satisfying $(-t_i)^n = -\tau_i$. Then, on account of Theorem 3, there is a unique curve $\gamma$ of type $B_n$ given by a line-equation

$$\Gamma\colon\ \ x - c + \sum_{k=1}^{n-1}(-1)^k\binom{n}{k}a_k t^k + (-1)^n(\overline{x} - \overline{c})t^n = 0 \qquad (24)$$

where $a_{n-k} = \overline{a_k}$ for all $k$, such that $\Gamma$ is tangent to line $L_i$ for parameter value $t_i$ $(i = 0, 1, \ldots, n)$. Line $L_i$ then is given by the line-equation

$$L_i\colon\ \ x - c + \sum_{k=1}^{n-1}(-1)^k\binom{n}{k}a_k t_i^k + (-1)^n(\overline{x} - \overline{c})t_i^n = 0. \qquad (25)$$

Since the equation of line $L_i$ is also given by $x - b_i - \tau_i\overline{x} = 0$, the $(n+1)$-tuple $(c, a_1, \ldots, \overline{a_1}, \overline{c})$ is the unique solution of the non-singular system (23) of $n+1$ linear equations in the $n+1$ unknown $z_0, \ldots, z_n$, which we repeat here:

$$b_i = z_0 - \sum_{k=1}^{n-1} (-1)^k \binom{n}{k} t_i^k z_k + (-1)^n t_i^n z_n.$$

Omitting line $L_0$ means omitting the first equation from this system, resulting in an $n \times (n+1)$ system of rank $n$. With the notation

$$p(t, z_0, \ldots, z_n) = z_0 - \sum_{k=1}^{n-1} (-1)^k \binom{n}{k} z_1 t^k + (-1)^n z_n t^n \quad (26)$$

this system may be written as

$$p(t_i, z_0, \ldots, z_n) = b_i \quad (i = 1, \ldots, n). \quad (27)$$

The solution set of this system is complex one-dimensional. The $(n+1)$-tuple $(z_0, z_1, \ldots, z_{n-1}, z_n) = (c, a_1, \ldots, \overline{a_1}, \overline{c})$ is a particular solution satisfying the additional condition

$$z_{n-k} = \overline{z_k} \quad (k = 0, \ldots, n).$$

Let $(v_0, \ldots, v_n)$ be any other solution of (27) satisfying

$$v_{n-k} = \overline{v_k} \quad (k = 0, \ldots, n).$$

Then the difference

$$(w_0, \ldots, w_n) = (c, a_1, \ldots, \overline{a_1}, \overline{c}) - (v_0, v_1, \ldots, v_{n-1}, v_n)$$

is a solution of the homogeneous system

$$p(t_i, w_0, \ldots, w_n) = 0 \quad (i = 1, \ldots, n) \quad (28)$$

satisfying the additional condition $w_{n-k} = \overline{w_k}$ for $k = 0, \ldots, n$.

For any solution $(w_0, \ldots, w_n)$ of (28), the function $p(t, w_0, \ldots, w_n)$ is a polynomial in $t$ of degree $n$ with zeros $t = t_1, \ldots, t = t_n$, so for some $\lambda \neq 0$,

$$\begin{aligned} p(t, w_0, \ldots, w_n) &= \lambda(t - t_1) \cdots (t - t_n) \\ &= \lambda(t^n - \sigma_1 t^{n-1} \\ &\quad + \cdots + (-1)^n \sigma_n) \end{aligned}$$

must hold, where the $\sigma_k$ are the symmetric functions defined by

$$\begin{aligned} \sigma_1 &= t_1 + t_2 + \cdots + t_n, \\ \sigma_2 &= t_1 t_2 + \cdots + t_{n-1} t_n, \\ \sigma_3 &= t_1 t_2 t_3 + \cdots + t_{n-2} t_{n-1} t_n, \\ &\vdots \\ \sigma_n &= t_1 t_2 \cdots t_n. \end{aligned}$$

Note that $\sigma_{n-k} = \overline{\sigma_k} \sigma_n$ for all $k = 1, \ldots, n-1$. Equating coefficients yields

$$\begin{aligned} w_0 &= \lambda(-1)^n \sigma_n, \\ -(-1)^1 \binom{n}{1} w_1 &= \lambda(-1)^{n-1} \sigma_{n-1}, \\ -(-1)^2 \binom{n}{2} w_2 &= \lambda(-1)^{n-2} \sigma_{n-2}, \\ &\vdots \\ -(-1)^{n-1} \binom{n}{n-1} w_{n-1} &= \lambda(-1) \sigma_1, \\ (-1)^n w_n &= \lambda, \end{aligned}$$

so

$$\begin{aligned} w_0 &= \sigma_n w_n \text{ and} \\ w_k &= -(-1)^n \lambda \sigma_{n-k} \binom{n}{k}^{-1} \\ &= -(-1)^n \lambda \overline{\sigma_k} \sigma_n \binom{n}{k}^{-1} \end{aligned}$$

for $k = 1, \ldots, n-1$. Hence, if and only if $\lambda$ is chosen such that

$$\overline{\lambda} = \lambda \sigma_n \quad (29)$$

we have

$$\begin{aligned} \overline{w_0} &= w_n \text{ and} \\ \overline{w_k} &= -(-1)^n \overline{\lambda} \sigma_k \overline{\sigma_n} \binom{n}{k}^{-1} \\ &= -(-1)^n \lambda \sigma_k \binom{n}{n-k}^{-1} \\ &= w_{n-k}. \end{aligned}$$

Equation (29) is a self-conjugate equation defining a line through the origin in the Argand plane. For any $\lambda$ on this line, let

$$\begin{aligned} c' &= c + w_0 = c + \lambda(-1)^n \sigma_n, \\ a'_k &= a_k + w_k = a_k - \lambda(-1)^n \sigma_{n-k} \binom{n}{k}^{-1}. \end{aligned}$$

Then $a'_{n-k} = \overline{a'_k}$ for all $k = 1, \ldots n-1$, so the self-conjugate equation

$$x - c' + \sum_{k=1}^{n-1} (-1)^k \binom{n}{k} a'_k t^k + (-1)^n (\overline{x} - \overline{c'}) t^n = 0$$

defines an $n$-curve $\Gamma_\lambda$, which for the parameter values $t_1, \ldots, t_n$ is tangent to the lines $L_1, \ldots, L_n$, respectively.

The locus of the centers $c'$ of these inscribed curves $\Gamma_\lambda$ consists of the line through $c$ with clinant

$$\frac{c' - c}{\overline{c'} - \overline{c}} = \frac{\lambda(-1)^n \sigma_n}{\overline{\lambda}(-1)^n \overline{\sigma_n}} = \sigma_n.$$

This line, with line-equation

$$L: \quad x - \sigma_n \overline{x} = c - \sigma_n \overline{c}$$

is called an *axis* of the system $L_1, \ldots, L_n$ of $n$ lines. The right-hand side of this equation equals the image of the origin upon reflection in $L$ (cf. equation (3)). Therefore, while $c$ depends on the lines $L_0, L_1, \ldots, L_n$, the expression $c - \sigma_n \overline{c}$ only depends on the lines $L_1, \ldots, L_n$ (and the chosen tangent parameter values $t_1, \ldots, t_n$). Morley ([3], p. 468) showed that

$$c - \sigma_n \overline{c} = (-1)^n \sum^n \frac{b_1 t_2 t_3 \cdots t_n}{(t_2 - t_1) \cdots (t_n - t_1)}$$

where in the summation the indices $1, \ldots, n$ should be permuted cyclically. We leave it as a challenge to the reader to verify this result.

It follows that the axis $L$ in terms of the images $b_i$ of the origin upon reflection in the lines $L_i$ and the chosen tangent parameter values $t_i$, is given by the self-conjugate equation

$$L: \quad x - \sigma_n \overline{x} = (-1)^n \sum^n \frac{b_1 t_2 t_3 \cdots t_n}{(t_2 - t_1) \cdots (t_n - t_1)}. \quad (30)$$

There are $n$ choices for each of the parameter values $t_i = -\sqrt[n]{-\tau_i}$ for $i = 1, \ldots, n$, each yielding one axis, so at first sight there are $n^n$ axes at most. However, multiplying each $t_i$ by the same factor $\omega^k$, where $\omega = e^{2\pi i/n}$, yields the same axis, so there are at most $n^{n-1}$ axes. The axes constitute the locus of the centers of inscribed $n$-curves.

Since the clinant of an axis equals $\sigma_n = t_1 \cdots t_n = \sqrt[n]{\tau_1 \cdots \tau_n}$, the axes occur in $n$ equispaced directions. Furthermore, multiplying one parameter value $t_i$ by the factor $\omega^k$ for $k = 0, \ldots, n-1$ while keeping the other parameter values $t_j$ the same, yields $n$ distinct clinants, so in each direction the number of axes is the same. □

For $n \geq 3$ it is possible that one or more of the lines $L_1, \ldots, L_n$ is a *double tangent line* of an inscribed $n$-curve $\Gamma$ (cf. Figures 3 and 4). For instance, if $L_1$ is a double tangent line then there are two distinct parameter values $t_1$ and $t'_1$ for which $L_1$ is tangent to $\Gamma$. Then, necessarily, these parameter values must differ by some factor $\omega^k \neq 1$, since $(-t_1)^n = (-t'_1)^n = -\tau_1$. The center $c$ of $\Gamma$ then is the intersection of two axes with clinants $\sigma_n = t_1 t_2 \cdots t_n$ and $\sigma'_n = t'_1 t_2 \cdots t_n$. Therefore, in general, the center of an inscribed $n$-curve with multiple tangent lines among the given lines $L_1, \ldots, L_n$ is the intersection of at least two axes.

Morley seems to claim that the converse is also true: if two axes intersect in a point $c$, then $c$ is the center of an inscribed $n$-curve for which (at least) one of the lines $L_1, \ldots, L_n$ is a multiple tangent line. He writes [3, p. 469]: *The curves $C^n$ which touch $n$ lines fall then into $n^{n-1}$ discrete systems. The transition from one system to another is when the center falls on two axes. One of the $n$ lines then is a double line of $C^n$.* (Morley denotes $n$-curves by $C^n$.) For $n = 3$ this is true; it follows, e.g., from the results in Van de Craats and Brinkhuis [1]. But, as Henk Pijls pointed out, a proof for $n > 3$ still is lacking. Morley apparently assumes that the center determines the inscribed $n$-curve uniquely. Then, indeed, it

**Morley, Miquel and Jiang Zemin**

Theorem 1, which I termed 'Morley's five circles theorem', is sometimes simply referred to as 'The five circles theorem'. Although I didn't make a deep search on its origin, I decided to name it after Morley, since it figures prominently on page 265 of his *Inversive Geometry* [4].

There are other related 'five circles theorems'. Probably the oldest one was published in 1838 by Auguste Miquel (1816–1851) in his article 'Théorèmes de géométrie'. In a slightly adapted notation, an English translation of his five circles theorem reads as follows:

**Theorem III.** *Let be given a pentagon $A_1A_2A_3A_4A_5$ with sides that are elongated to their mutual intersection points $B_1 = A_5A_1 \times A_2A_3$, $B_2 = A_1A_2 \times A_3A_4 \dots$. Construct the circumscribed circles of the five triangles $A_1B_1A_2, A_2B_2A_3, \dots$, formed by a side of the pentagon and the adjacent elongated sides. Then I say that the five other intersections of consecutive circles are concyclic.* [2, p. 486].

At first sight, this seems to be a kind of converse to Morley's five circles theorem, but Miquel's theorem doesn't mention the centers of the five circles. Although some internet sources claim that these centers necessarily must lie on the Miquel circle, it is easily verified by counterexamples that this is not true in general. Recently, Miquel's five circles theorem attracted attention when in 1999 Jiang Zemin, at that time the president of the People's Republic of China, reportedly put it as a challenge to students during a visit to Macau, and also in 2002, when he attended the International Congres of Mathematicians in Beijing.

would follow that the intersection of two axes is the center of an inscribed $n$-curve with a multiple tangent line among the given lines. However, we doubt whether this assumption is true in general.

For $n = 2$ the inscribed 2-curves are circles touching two given intersecting lines $L_1$ and $L_2$, and the $2^{2-1} = 2$ axes are the two angle bisectors.

For $n = 3$ the inscribed 3-curves are cardioids touching three given lines, which leads us back to Morley's famous trisector theorem. If the three lines are the extended sides of a triangle, then there are $3^{3-1} = 9$ axes, 3 axes in each of 3 equispaced directions, see [1] for more details on this case. If the three given lines are concurrent, then there are only 3 axes, with clinants that are the three geometric means of the clinants of the given lines, each axis passing through the common point of the given lines.

This brings to an end our introduction to Morley's treatment of curves of type $B_n$. The interested reader is referred to Morley's article [3] and his book [4] for other intriguing and challenging results in this field. Let us finish this paper by quoting the following lines from Morley's 1929 article ([3], p. 469):

> "If we apply the theory of this section to a triangle $abc$, we obtain as the locus of the centers of inscribed cardioids three sets of three parallel lines, forming equilateral triangles. The vertices of the triangles are the centers of the cardioids which touch a side (say $bc$) of the given triangle twice. If $x_0$ is such a center, then the angle $x_0bc$ is a third of the angle $abc$. For $x_0b$ is an axis of the 3 lines $ab$ and $bc$ twice. Thus if we take the interior trisectors of the angles, the points where those adjacent to a side meet form an equilateral triangle."

In a footnote he adds: "This theorem, which I obtained in this way long ago, has excited much interest."

## References

1 Jan van de Craats and Jan Brinkhuis, Cardioids and Morley's trisector theorem, *Nieuw Archief voor Wiskunde* 5/18(1) (2017), 25–33.

2 A. Miquel, Théorèmes de géométrie, *J. de mathématiques pures et appliquées* 1 (1838), 485–487. http://gallica.bnf.fr/ark:/12148/bpt6k1638m/f494.image

3 F. Morley, Extensions of Clifford's chain-theorem, *Amer. J. of Math.* 51 (1929), 465–472.

4 Frank Morley and F.V. Morley, *Inversive Geometry*, Ginn, Boston, 1933. (Reissued by Chelsea, 1954 and by Dover Books on Mathematics, 2013.)

Recommended internet page:

Alexander Bogomolny, *Morley's Miracle* from Interactive Mathematics Miscellany and Puzzles, http://www.cut-the-knot.org/triangle/Morley/index.shtml.