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Trip to the proof

Mixing, wine, and serendipity

Mixing for measure-preserving group actions is a fundamental notion in ergodic theory, with different phenomena arising for different acting groups. In 1993, Klaus Schmidt and Tom Ward proved that 2-mixing implies mixing of all orders for actions by commuting automorphisms of connected groups. Tom Ward explains how he became interested in this problem, and describes how a chance encounter with a paper on a seemingly unrelated problem in number theory played a key role.

During the 1980's, while I was a graduate student at Warwick University under the supervision of Klaus Schmidt, a specific kind of algebraic dynamical system was emerging as a surprisingly rich and relatively unexplored field. In hindsight, a small shift in how a key example constructed by Ledrappier is thought of might have predicted some of this, but prediction with the benefit of hindsight is a little too easy.

Mixing is a mathematical version of the idea of, well, mixing. If two ingredients of a cocktail are poured carefully into a glass — so carefully that perhaps they form individual layers — then the action of a stirrer is 'mixing' if after some time every mouthful tastes the same. That is, every part of the glass has the ingredients in the same proportion up to a negligible error. This becomes a mathematical concept by noticing that the volume may be viewed as a measure on the space consisting of the contents of the glass, and the action of the stirrer might be thought of as iteration of a map that preserves that measure (unless it is being stirred using a straw, and the person stirring is taking a crafty sip every now and then). Avoiding all the interesting and subtle physical and chemical issues involved — particularly egregious in the circumstances — we might as well assume the action of stirring is invertible, and for mathematicians the resulting structure of a measure-preserving action of the integers might as well be an action of any group. Having no wish to trip up on any measure theory, let's say that the group is countable. So here is mixing: if a countable group G acts by transformations preserving a measure μ on a probability space, then it is called mixing if $\mu(A\cap gB)$ converges to $\mu(A)\,\mu(B)$ as g 'goes to infinity' in G. And why not be ambitious? Mixing on $k+1$ sets (or mixing of order k) means that for any measurable sets $A_0,\ldots,A_k$ the measure

$$\mu(A_0\cap g_1A_1\cap\cdots\cap g_kA_k)$$

of the intersection converges to $\prod_{j=0}^{k}\mu(A_j)$ as the group elements g_j go to infinity and move apart from each other. So here is a mathematical question: given a measure-preserving action of a countable group, determine if it is mixing on k sets for a given k. When $G=\mathbb{Z}$ it is a long-standing question of Rokhlin as to whether mixing on 2 sets forces mixing on 3 sets.

Which brings us to Ledrappier's example [1] (simplified for convenience from his harmonic condition example): let X be the subset of $\{0,1\}^{\mathbb{Z}^2}$ consisting of the points x satisfying $x_{s,t}=x_{s+1,t}+x_{s,t+1}$ modulo 2 for every $(s,t)\in\mathbb{Z}^2$. This is a compact group, and the shift in $\mathbb{Z}^2$ defines an action of $\mathbb{Z}^2$ that preserves the natural Haar measure. The system is easily shown to be mixing on 2 sets, but the fact that the relation $x_{s,t}=x_{s+2^n,t}+x_{s,t+2^n}$ modulo 2 holds for all $n\geq 1$ (a direct consequence of the properties of the Frobenius $a\mapsto a^2$ modulo 2 under iteration) forces a correlation between triples of sets separated by arbitrarily large distances — failure of mixing on 3 sets. Ledrappier also pointed out that any system like this built from automorphisms of compact groups has a property called 'Lebesgue spectrum'. A productive shift in perspective is to think of this system as the dual group of the module $\mathbb{Z}[u_1^{\pm1},u_2^{\pm1}]/\langle 1+u_1+u_2\rangle$, with the action of $(a,b)\in\mathbb{Z}^2$ dual to multiplication by $u_1^a u_2^b$. Thus a version of the mixing question becomes this: describe the mixing properties of such a system built from a module M over the ring $R=\mathbb{Z}[u_1^{\pm1},\ldots,u_d^{\pm1}]$ in terms of properties of the module M — in the certain knowledge that the answer is non-trivial because it is for Ledrappier's example.

Work of Kitchens and Schmidt [4] probed the mixing properties of systems whose compact group is zero-dimensional, uncovering a complex collection of properties leading to many interesting questions. Schmidt [3] also showed that the way in which a 'shape' produced by the Frobenius automorphism witnesses failure of higher-order mixing as seen by Ledrappier could not take place if the compact group X is connected. That is, for a mixing $\mathbb{Z}^d$ action by automorphisms of a compact connected group, choosing the times $g_1,\ldots,g_k$ to be dilates of a fixed shape in $\mathbb{Z}^d$ would never show failure of mixing, raising the question: for these connected systems, does mixing imply mixing of all orders?

By 1991 I was working at Ohio State University, and we were notified that some duplicate journals were being discarded. As life was then full of time for mathematics, I went into the basement and leafed

Doug Lind, Tom Ward and Klaus Schmidt at a Lorentz Center workshop in 2014

through piles of journals in recycling bins, tearing out any articles that looked vaguely interesting. I piled these up, and left to attend a workshop at CIRM in Luminy.

In that beautiful place, Klaus Schmidt reminded Doug Lind and me of this open problem over a splendid meal. Perhaps with the assistance of the generous provision of wine, I became sure that I had an argument, essentially using the Lebesgue spectrum property, that proved mixing of all orders for these connected systems. Not for the first, and not for the last, time, Klaus let me whitter on for some time as we walked under the pine trees before politely pointing out that my suggested argument applied unchanged to Ledrappier's example.

Flying back to Columbus, the problem was firmly in my mind. The '$\times 2, \times 3$' system, itself studied for other reasons, was the natural start. Ledrappier's salutary example showed that the result sought really couldn't come from the familiar toolbox of spectral or entropy methods. Via Fourier analysis of indicator functions of sets, it seemed to come down to this: what can you say about solutions of $\sum_{j=1}^{k} a_j x_j = 1$ in a number field, where the variables x_j come from a finitely-generated multiplicative subgroup? For '$\times 2, \times 3$' the field would be $\mathbb{Q}$, and the multiplicative subgroup $\{2^a 3^b \mid a, b \in \mathbb{Z}\}$. Failure of mixing of all orders seemed roughly equivalent to equations of this shape having too many — infinitely many — non-trivially different solutions. Trivially different solutions abound if a sub-sum vanishes, because that vanishing sub-sum can be scaled by powers of 2 and 3 arbitrarily.

After a few days back in Columbus, I sorted through the pile of torn-out papers on my desk. One was a (then) recent paper of Schlickewei [2] with a form of 'S-unit theorem'. For the finite-dimensional case at hand (it turned out later that the topological dimension of the compact group X plays a role) a simple reduction argument was suddenly completely clear. If a $\mathbb{Z}^d$-action by automorphisms of a compact connected abelian group fails to be mixing on $k \geq 3$ sets, then by the Fourier analysis argument there is a linear equation with k terms over a field of characteristic zero that has infinitely many distinct solutions lying in a multiplicative group with d generators (corresponding to the automorphisms defining the action). By the S-unit theorem, this is impossible unless infinitely many of them come from a vanishing sub-sum: a linear equation with $j < k$ terms. But finding infinitely many solutions for that shorter linear equation is a witness to failure of mixing on $j < k$ sets. Thus mixing on 2 sets implies mixing of all orders. Some cleaning up was needed — algebra to reduce to cyclic modules, and a more subtle process needed to deal with infinite-dimensional compact groups which do not readily permit the translation into statements in number fields — but this quickly led to the proof of the full case with Klaus Schmidt [5].

There are some lessons to take from this strange coincidence and happy resolution. Certainly ideas produced under the influence of wine may eventually face the sobering reality of counter-examples — but can be motivating nonetheless. More importantly, Tramezzino's tale *Peregrinaggio di tre giovani figliuoli del re di Serendippo* in which 'accidents and sagacity' play such a role still has something for us. The rush of modern academic life, the growing use of online journals and their sophisticated and well-intentioned nudging towards related articles, the demarcation of subject areas, the overwhelming growth in the volume of the mathematical literature — all make the benefits of serendipity less easy to access. If you have the good fortune of time to spend on mathematics, spend some of it on the not 'suggested article', on the articles that readers are not 'also reading', and on items with the wrong subject classification codes.

References

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