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Computational Conley theory

In this article William Kalies and Robert Vandervorst describe Conley theory. Conley theory studies the dichotomy between gradient-like and non-gradient-like (recurrent) behavior of dynamical systems. It is a topological theory and one is compelled to ask the question whether this theory is computable. Numerical simulation has proved to be a common technique for analyzing dynamics, but does not give a complete picture of global behavior. By discretizing a dynamical system in time and space one can build a combinatorial dynamical system which carries dynamical information of the original system within a given resolution. The novelty of computational Conley theory is the combination of algebra, topology and combinatorics which produces computable tools that can be used to prove rigorous statements about global dynamic behavior.

Many evolutionary processes in physics, engineering, biology, et cetera are modeled by dynamical systems. Examples of dynamical systems range from very simple models such as the motion of a pendulum to extremely complex systems such as for instance meteorological models. Dynamical systems are mostly defined via systems of differential equations, but also as iterations of continuous maps. In practice these systems are too complex to study without any computational tools. A first insight into the workings of a dynamical system is often provided by numerical simulation in order to understand for example long term behavior of a system, which has been extremely successful in the application of dynamical systems as a prediction tool. Such analysis does not give much insight into global behavior in general, nor does it provide rigorous results about the dynamical structure of a system. Computational Conley theory makes various aspects of global dynamics computable using algebra, combinatorics and algebraic topology.

Going beyond local analysis such as linearized behavior of equilibrium points, or periodic orbits, is often very hard for even low dimensional systems let alone complex high dimensional systems. The development of the concepts of Conley theory into computable tools is of great benefit to the study of global dynamics. One of the key theorems in Conley theory is *Conley's decomposition theorem* which states that global dynamics can be categorized in (chain-)recurrent and gradient-like dynamics. For example consider the motion of a natural pendulum. As the pendulum swings, it cannot return to its original position with the same velocity because it loses energy due to fric-

tion — gradient-like dynamics, cf. Figure 1. In the absence of friction the system returns to its initial state, and its motion is periodic. In this case the system is said to exhibit recurrence. Understanding this dichotomy between (chain-)recurrent and nonrecurrent or gradient-like behavior is central to the study of dynamical systems and an important aspect of Conley theory. The theory was initially developed as a generalization of Morse theory for flows on smooth manifolds, see [4]. Conley theory also addresses the dependence of a system on parameters which is a very natural question to ask from the point of view of applications. Over the past thirty years this theory has been extended to cover most types of (deterministic) dynamical systems such as flows, discrete time systems and infinite dimensional dynamical systems, cf. [20–24].

The chain-recurrent and gradient-like dynamics of a given system is computable within a given resolution. We will explain
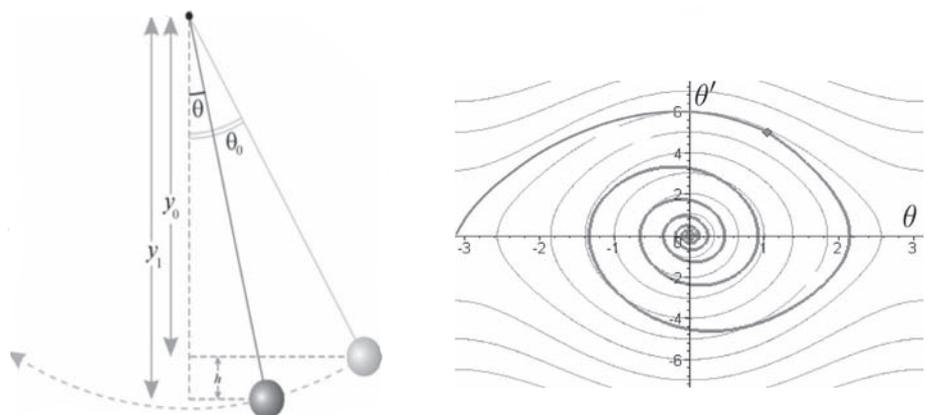
Figure 1 The phase portrait of a natural pendulum.

that these concepts can be alternatively defined so that they are more tangible for the purpose of rigorously computing them. This becomes a matter of understanding the key algebraic and combinatorial structures that play a role and their connection to the topology of the system. The new direction and novelty of this approach lies in the fact that the study of dynamical systems is reduced to the study of (finite) combinatorial dynamical systems in combination with (computable) algebraic structures and invariants. Algebraic topology takes continuum objects and reduces them to algebraic objects which are typically finitely generated and hence can be manipulated by the computer. The algebra carries well understood information about the continuum structure. The methods described here build on ideas that have emerged over last decade due to many authors in which combinatorialization and algebraic topology are the key ingredients. The first steps in this direction have proved very promising. We mention the early work of Eidenschink and Boczko, Kalies and Mischaikow [3, 10, 18], see also [7–9, 11, 12, 19, 24] and [14–17].

On a more philosophical note modern science is all about collecting and processing data. Most of the data comes from dynamical processes that are traditionally modeled in terms of continua, i.e. differential equations, manifolds, differentiable maps, et cetera. The process of collecting data is inherently a discretization. The ideas presented here may have major impact as science moves more and more towards data based representations of science and away from analytical model based representations of science. These developments make a computational Conley theory a valuable asset as a tool in applied dynamical systems. A successful computational theory will allow us to go beyond studying dynamical systems in low dimensions and will open the door to better understanding complex systems via rigorous statements.

Dynamical systems and algebraic structures
In mathematical terms a dynamical system consists of the following ingredients; a state space, or phase space X (metric space) and a family of maps φ_t, parametrized by a time parameter t, mapping X (on)to itself. For instance for the physical pendulum the state space consists of

the rotation angles and is an example of a one-dimensional phase space. In more complicated systems such as biological or meteorological models the state space may be high dimensional. The time parameter t is either discrete, $t \in \mathbb{Z}^+$, or continuous, $t \in \mathbb{R}^+$. Dynamical systems describe deterministic processes; for an initial state x the state $\varphi_t(x)$ at time t is determined entirely by the initial state x. This is captured by the (semi-)group property for φ_t:

$$\begin{cases} \varphi_0(x) = x, \\ \varphi_{s+t}(x) = \varphi_s(\varphi_t(x)) \quad \text{for all } s,t \geq 0, \end{cases}$$

and for all $x \in X$. For many aspects of Conley theory it is not important whether the time variable is discrete or continuous. In order to keep exposition transparent we restrict to discrete time dynamical systems. If $t \in \mathbb{Z}^+$, then

$$\varphi_t = f^t = \underbrace{f \circ \cdots \circ f}_{t \text{ times}}, \quad f = \varphi_1,$$

where f is called the *generator*, or *generating map* and the discrete time dynamical system is denoted by (X, f). We emphasize that the generator $f = \varphi_1$ is not necessarily invertible! (For continuous time this allows for semi-flows.)

Robust structures
Consider the two-dimensional state space $X = [0,1] \times [0,1]$ with $x = (\xi, \eta) \in X$ and the mapping $f : X \to X$ given by

$$f(\xi, \eta) = \left(\frac{\xi}{2-\xi}, \frac{\eta}{2-\eta} \right), \qquad (1)$$

Figure 2 depicts the dynamics of f. We will refer to as Example 1.

We first introduce terminology to describe asymptotic behavior of orbits. A *complete orbit* through x is denoted by γ_x and the forward and backward portions are denoted by γ_x^+ and γ_x^- respectively. The *omega-limit set* of a set $N \subset X$ is defined as

$$\omega(N) = \bigcap_{k \geq 0} \text{cl} \left(\bigcup_{n \geq k} f^n(N) \right),$$

and the *alpha-limit set* is defined as

$$\alpha(N) = \bigcap_{k \leq 0} \text{cl} \left(\bigcup_{n \leq k} f^n(N) \right).$$

The *orbital alpha-limit* and *orbital omega-limit* set are defined as $\alpha_o(\gamma_x^-) = \bigcap_{t \leq 0} \text{cl} \gamma_x((-\infty, t])$ and $\omega(x) = \omega(\{x\})$ respectively.

The square $X = [0,1] \times [0,1]$ in Example 1 is invariant, i.e. f maps X onto X. To be
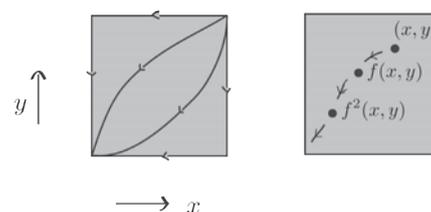
Figure 2 Complete orbits of f (left) and the iterates of f (right).

more precise a subset $S \subset X$ is *invariant* for the dynamical system (X, f) if $f(S) = S$. In the above example many invariant sets can be found. For instance, the corner points, the boundary, et cetera. An invariant set $A \subset X$ is called an *attractor* if there exist a compact neighborhood $N \subset X$ containing A such that $f(N) \subset N$, $f^k(N) \subset \text{int}(N)$ for some $k > 0$ and $A = \omega(N) = \text{Inv}(N, f)$, where $\text{Inv}(N, f)$ indicates the maximal invariant set inside N. The set $N \supset A$ is called a *trapping region* for A. The choices $N = \varnothing$ and $N = X$ are trivial trapping regions. Examples of attractors for are $A_1 = (0,0)$, $A_2 = \{(\xi, 0) \mid \xi \in [0,1]\}$, and $A_3 = \{(0, \eta) \mid \eta \in [0,1]\}$. Also the empty set and the state space X are attractors in Example 1. Dual to an attractor A is its *dual repeller* defined by

$$A^* = \{x \in X \mid \omega(x) \cap A = \varnothing\}.$$

For example $A_2^* = \{(\xi, 1) \mid \xi \in [0,1]\}$, $X^* = \varnothing$ and $\varnothing^* = X$. A dual repeller may also be defined as $A^* = \alpha(N^c)$ where N^c is called a repelling region. A pair (A, A^*) is called an *attractor-repeller pair* for (X, f) and is a robust structure. Attractor-repeller pairs give a very rough insight into the dynamics and they are not unique in general. Example 1 shows six different attractor-repeller pairs. Given an attractor-repeller pair (A, A^*) then for any $x \in X \setminus \{A \cup A^*\}$ it holds that $\alpha_o(\gamma_x^-) \in A^*$ and $\omega(x) \in A$ for any complete orbit γ_x through x. The idea of decomposing the dynamics into invariant sets and points which converge to theses sets can be further generalized and leads to the notion of Morse representation.
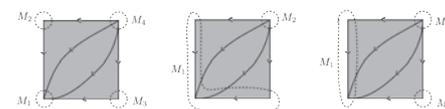
Figure 3 Three Morse representations. The finest representation consisting of equilibrium points (left), an attractor-repeller pair (middle), and a 3-set Morse representation (right).

Definition 1. A finite poset $(\mathsf{M}, \leq)$ consisting of non-empty, pairwise disjoint, compact invariant subsets of X is called a *Morse representation* for (X, f) if for every $x \in X \backslash (\bigcup_{M \in \mathsf{M}} M)$, there exist $M' < M''$, with $M', M'' \in \mathsf{M}$, such that

$$\omega(x) \subset M' \text{ and } \alpha_{\mathrm{o}}(\gamma_x^-) \subset M'',$$

for any complete orbit γ_x through x.

Figure 4 The four non-trivial attractors in Example 1 (left), and its lattice structure (right).

(We use the terminology *poset* to indicate a partially ordered set.)

The sets $M \in \mathsf{M}$ are called *Morse sets*. In general Morse sets occur as intersections of attractors and repellers, i.e.

$$M = A \cap A'^*,$$

for some attractor A and repeller A'^* dual to some attractor A'. In the next subsection we will explain more of the systematics behind Morse representations.

Via a Morse representation $(\mathsf{M}, \leq)$ *we obtain a combinatorial description — a poset — of the global structure of a dynamical system within a given resolution.*

Morse representations can be further refined by finding attractor-repeller pair decompositions (or Morse representations) of Morse sets, see Figure 3 middle and right.

For some systems this process terminates after finitely many steps and a finest Morse representation exists as in the above example. If not, the intersection $\bigcap (A \cup A^*)$, ranging over all attractors in the system, leads the so-called chain-recurrent set $\mathsf{R} \subset X$. This set is a countable union of chain-connected components R_i which are defined as follows. Two points $x, x' \in \mathsf{R}$ lie in the same connected component R_i if for any attractor-repeller pair (A, A^*) either both $x, x' \in A$ or both $x, x' \in A^*$. Conley's decomposition theorem states that the dynamics outside R is gradient-like. The term chain-recurrence is inherited from its original definition via ε-chain recurrence. The latter states that the components R_i have the property that each point $x \in \mathsf{R}_i$ comes back to itself under the dynamics arbitrary close allowing small errors — ε-chains. The chain-recurrent set R is obtained as a limit. From the point of view of computation this definition is not very practical, cf. [14].

Lattices and Morse representations
The set of attractors (and repellers) in a dynamical system has the algebraic structure of a distributive lattice, cf. [15, 21]. A *distributive lattice* is a set L equipped with
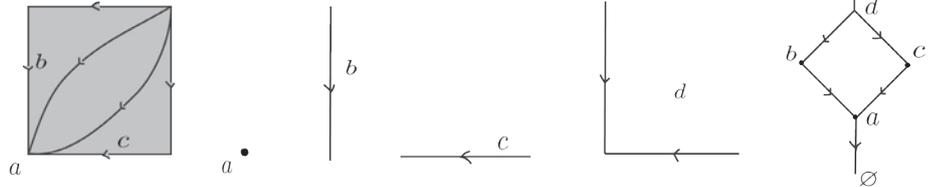
two binary operation $\vee$ and $\wedge$. The operations $\vee$ and $\wedge$ play the role of union and intersection and they satisfy the same axioms as union and intersection for subsets. A distributive lattice has a poset structure and $a \leq b$ if and only if $b = a \vee b$ (or equivalently $a = a \wedge b$). Let $\mathsf{Att}(X, f)$ be the set of all attractors of a dynamical system (X, f) and define the binary operations

$$A \vee A' = A \cup A', \quad A \wedge A' = \mathrm{Inv}(A \cap A', f).$$

With respect to these binary operations $\mathsf{Att}(X, f)$ is a bounded distributive lattice, see [15]. In general $\varnothing$ and $\omega(X)$ play the role of 0 and 1, respectively. (The term *bounded* means that $\inf(\mathsf{Att})$ and $\sup(\mathsf{Att})$ exist. As a matter of fact $\inf(\mathsf{Att}) = \varnothing$ and $\sup(\mathsf{Att}) = \omega(X)$.) The associate repellers form an (anti-)isomorphic lattice called $\mathsf{Rep}(X, f)$ with binary relations $\vee = \cup$ and $\wedge = \cap$. The trapping regions and repelling regions are also bounded distributive lattices with binary relations intersection and union and are denoted by $\mathsf{TrapR}(X, f)$ and $\mathsf{RepR}(x, f)$, respectively. Duality between the lattices is expressed in the following fundamental commuting diagram:

$$
\begin{array}{ccc}
\mathsf{TrapR}(X, f) & \xleftarrow{\;c\;} & \mathsf{RepR}(X, f) \\
\omega \downarrow & & \downarrow \alpha \\
\mathsf{Att}(X, f) & \xleftarrow{\;*\;} & \mathsf{Rep}(X, f)
\end{array}
\qquad (2)
$$

where both c and $*$ are involutions, and are the respective duality mappings. Figure 4 shows the attractors in Example 1 together with the lattice structure of $\mathsf{Att}(X, f)$. The lattice structure of $\mathsf{Att}(X, f)$ is intimately related to the poset structure of Morse representations. To explain this relation we introduce some elementary notions of poset theory.

Given a finite poset P then a *down-set* I consists of those elements $p \in \mathsf{P}$ such that $q \leq p$ implies that $q \in I$. Denote the set of all down-sets of P by $\mathsf{O}(\mathsf{P})$ which is a finite distributive lattice with respect

to $\vee = \cup$ and $\wedge = \cap$. Beside the down-sets one can also define *up-sets* J which consists of those elements $p \in \mathsf{P}$ such that $q \geq p$ implies that $q \in J$. The up-sets form the distributive lattice $\mathsf{U}(\mathsf{P})$ dual to $\mathsf{O}(\mathsf{P})$ via set-complement. The *join-irreducible elements* in a finite distributive lattice L are the elements that are preceded by a unique predecessor, cf. Figure 5. A classical result by G. Birkhoff [2] provides a representation theorem for finite distributive lattices which is formulated as follows. Let L be a finite distributive lattice and let P be a finite poset. By $\mathsf{J}(\mathsf{L})$ we denote the poset of join-irreducible elements where the order is defined by set-inclusion. Then,

$$\mathsf{L} \cong \mathsf{O}(\mathsf{J}(\mathsf{L})), \quad \mathsf{P} \cong \mathsf{J}(\mathsf{O}(\mathsf{P})). \qquad (3)$$

Figure 5 gives a schematic account of Birkhoff's theorem in the case of Example 1. The operations O and J are contravariant functors and lattice monomorphisms $f : \mathsf{L} \rightarrowtail \mathsf{K}$ are equivalent to order-surjections $\mathsf{J}(f) : \mathsf{J}(\mathsf{K}) \twoheadrightarrow \mathsf{J}(\mathsf{L})$, and similarly lattice epimorphisms are equivalent to order-embeddings. The same correspondence can also be formulated via the O functor using Birkhoff's representation theorem, cf. [6].

Morse representations and finite sublattices of attractors in $\mathsf{Att}(X, f)$ are equivalent as a consequence of Birkhoff's representation theorem. This relation can be seen as follows, cf. Figure 5. Let $\mathsf{A} \subset \mathsf{Att}(X, f)$ be a finite sublattice. For every join-irreducible attractor $A \in \mathsf{J}(\mathsf{A})$ let $\overleftarrow{A} \prec A$ be the unique predecessor and define $M = A \cap \overleftarrow{A}^*$. It can be shown that the set $\mathsf{M}(\mathsf{A}) = \{A \cap \overleftarrow{A}^*\}$, with $A \in \mathsf{J}(\mathsf{A})$ and the order-relation induced by $\mathsf{J}(\mathsf{A})$, is a Morse representation and the sets $M = A \cap \overleftarrow{A}^*$ are Morse sets. In general $M = A_I \cap A_J^*$, where the representation only depends on the set-difference $I \backslash J = \{M\}$, with $I, J \in \mathsf{O}(\mathsf{M})$. For the lattice of attractors $\mathsf{A} = \mathsf{Att}(X, f)$ in Example 1 we have $\mathsf{J}(\mathsf{A}) = \{a, b, c, X\}$ and the associated poset $\mathsf{M}(\mathsf{A}) = \{M_1, \ldots, M_4\}$ is the finest Morse representation in Example 1 consisting of the rest points. The above procedure
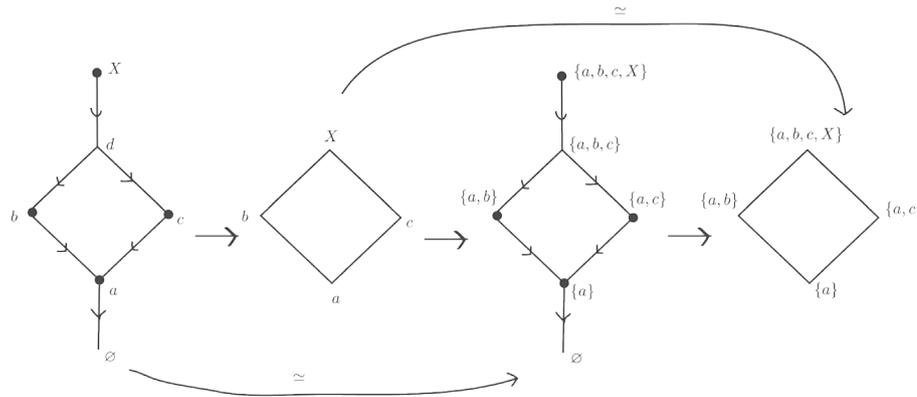
Figure 5 The join-irreducible elements form a poset J(A) and the down-sets of J form a lattice O(J(A)) which is isomorphic to A. The solid dots in the diagrams denote the join-irreducible elements.

yields an isomorphism between $(M(A), \leq)$ and $(J(A), \subseteq)$. Conversely, by Birkhoff's theorem the lattice of down-sets in $(M(A), \leq)$ is isomorphic to A. The isomorphism is quantified as follows. If $I \in O(M)$, then the sets $A_I = \bigcup_{M \in I} W^u(M)$ are attractors which build the lattice A. (The set $W^u(M)$ denotes the *unstable set* of M and is defined as: $W^u(M) = \{x \in X \mid \exists \gamma_x^- \ni \alpha_o(\gamma_x^-) \subset M\}$.)

Morse decompositions
Let $A \subset \text{Att}(X, f)$ be a finite sublattice (including $\varnothing$ and $\omega(X)$) and let $A^* \subset \text{Rep}(X, f)$ be the associated dual and let P be a finite poset such that $O(P)$ maps onto A (as a lattice homomorphism). Consider the commuting diagram

$$
\begin{array}{ccc}
O(P) & \xleftarrow{\ c\ } & U(P) \\
\downarrow & & \downarrow \\
A & \xleftarrow{\ *\ } & A^*
\end{array}
\qquad (4)
$$

Since Birkhoff's representation theorem is functorial the vertical lattice-surjections are dual to the order-injection $J(A) \hookrightarrow P$. Using the Morse representation for $J(A)$ as described in the previous section we obtain the order-injection $M(A) \hookrightarrow P$, which is referred to as a *Morse decomposition*.
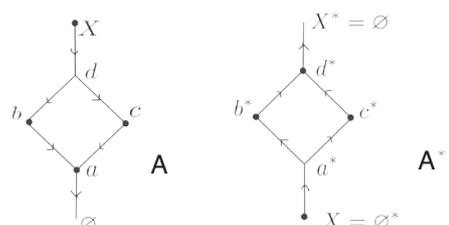
Figure 6 A lattice of attractors and the anti-isomorphic lattice of dual repellers.

The poset $(P, \leq)$ may be regarded as a combinatorial model for the dynamics of (X, f) if we interpret P as directed graph, or *digraph*. The order-injection $M(A) \hookrightarrow P$ signifies the dynamically relevant sets (Morse sets).

Consider an embedding $O(P) \hookrightarrow \text{TrapR}(X, f)$ and denote the associated sublattices (images) by N. If we define $\omega(N) = A \subset \text{Att}(X, f)$, then Diagram (2) yields

$$
\begin{array}{ccc}
N & \xleftarrow{\ c\ } & N^c \\
\omega \downarrow & & \downarrow \alpha \\
A & \xleftarrow{\ *\ } & A^*
\end{array}
\qquad M(A) \rightarrowtail T(N), \quad (5)
$$

where $T = T(N)$ is a representation of $J(N) \cong P$ via the identification: $T_p = N_I \cap N_J^c = N_I \setminus N_J$ which depends only on $I \setminus J = \{p\}$ — compare the construction of a Morse representation. The poset $T(N)$ is called a *Morse tiling* and the order-embedding $M(A) \hookrightarrow T(N)$ is called a *tesselated Morse decomposition*.

By construction $P \cong T(N)$ and the elements of $T(N)$, called *Morse tiles*, tesselate the space X. If the poset set structure of $T(N)$ is regarded as a digraph, then a tesselated Morse decomposition is a first example of a combinatorialization of a dynamical system, i.e. the order relation indicates where points in X may map under the dynamics of f. The embedding $M(A) \hookrightarrow T(N)$ marks the dynamically relevant tiles, i.e. the tiles that contain a Morse set.

The theory of Morse representations and decompositions as a combinatorial description of dynamics is a first motivation for the algebraic treatment of attractors via the theory of distributive lattices.

Combinatorial dynamics
The existence of a finite sublattice $N \subset \text{TrapR}(X, f)$ yields a tesselated Morse decomposition, cf. the previous subsection. If we start with a finite sublattice $A \subset \text{Att}(X, f)$ it is much harder to find a sublattice $N \subset \text{TrapR}(X, f)$ such that $\omega(N) = A$. The latter is called the lifting problem. In [15, Theorem 1.2] a lifting theorem of finite sublattices in $\text{Att}(X, f)$ is proved: For every finite sublattice $A \subset \text{Att}(X, f)$ there exists a finite sublattice $N = k(A) \subset \text{TrapR}(X, f)$, called a *lift*, such that $\omega(N) = A$. The lift $N := k(A) \subset \text{TrapR}(X, f)$ provides the Morse tiling $T(N) \cong J(A)$ as described in the previous section. Since finding Morse representations is one of the objectives we cannot a priori use them in order to combinatorialize dynamics. In the next subsection we describe an explicit procedure to combinatorialize a dynamical system.

Discretization of space
Let $\mathcal{X}$ be a finite 'labeling set' with labels $\xi \in \mathcal{X}$ and $|\xi| \subset X$ are subsets of X satisfying the following axioms:

1. $X = \bigcup_{\xi \in \mathcal{X}} |\xi|$;
2. $|\xi| = \text{cl}(\text{int}(|\xi|))$ for all $\xi \in \mathcal{X}$;
3. $|\xi| \cap \text{int}(|\eta|) = \varnothing$ for all $\xi \neq \eta \in \mathcal{X}$.

The set $\{|\xi|\}_{\xi \in \mathcal{X}}$ is called a *grid* on X and the elements $|\xi|$ are called *grid-elements*. The labeling set is also referred to as a grid in X. If we consider the Boolean algebra of *regular closed sets* on X, denoted $\text{RG}(X)$, then choosing a grid on X is equivalent to selecting a finite subalgebra in $\text{RG}(X)$. The latter is similar to the relation between attractor lattices and Morse representations. The subalgebra is the equivalent of the sublattice and the join-irreducible elements in the subalgebra are called the *atoms*. As with Morse tiles, the grid elements are constructed in the same way. The only difference is that $\mathcal{X}$ is a trivial poset by construction, i.e. no order relation, cf. [16].

In order to mimic the dynamics of (X, f) we define a multivalued mapping $\mathcal{F} : \mathcal{X} \rightrightarrows \mathcal{X}$ on $\mathcal{X}$. The multivalued mapping $\mathcal{F} : \mathcal{X} \rightrightarrows \mathcal{X}$ is linked to the dynamics of f via the condition

$$
f(|\xi|) \subset \text{int} |\mathcal{F}(\xi)|, \qquad (6)
$$

for all $\xi \in \mathcal{X}$, cf. [24]. Such multivalued mappings are called *outer approximations* for (X, f). The grid $\mathcal{X}$ together with a mul-

tivalued mapping $\mathcal{F}$ is called a *combinatorial dynamical system*; notation $(\mathcal{X}, \mathcal{F})$. If we regard the elements in $\mathcal{X}$ as vertices and the action of $\mathcal{F}$ by edges, then $(\mathcal{X}, \mathcal{F})$ may be regarded as a digraph; $\xi_i \in \mathcal{X}$ are vertices and e_{ij} is an edge from ξ_i to ξ_j if $\xi_j \in \mathcal{F}(\xi_i)$.

Combinatorial dynamical systems
Multivalued mappings regarded as combinatorial dynamical systems share many similarities with dynamical systems (X, f). Except for multivaluedness most of the dynamical concepts carry over to combinatorial systems. A subset of vertices $\mathcal{S} \subset \mathcal{X}$ is called *invariant* if $\mathcal{S} \subset \mathcal{F}(\mathcal{S})$ and $\mathcal{S} \subset \mathcal{F}^{-1}(\mathcal{S})$. A subset $\mathcal{A} \subset \mathcal{X}$ is called an *attractor* if $\mathcal{F}(\mathcal{A}) = \mathcal{A}$, and similarly a subset $\mathcal{R} \subset \mathcal{X}$ is called a *repeller* if $\mathcal{F}^{-1}(\mathcal{R}) = \mathcal{R}$. A *dual repeller* to an attractor $\mathcal{A}$ is defined as $\mathcal{A}^* = \boldsymbol{\alpha}(\mathcal{X} \backslash \mathcal{A})$. (Recall the definition of alpha and omega limit set form $(\mathcal{X}, \mathcal{F})$: $\boldsymbol{\omega}(\mathcal{U}) = \bigcap_{k \geq 0} \bigcup_{n \geq k} \mathcal{F}^n(\mathcal{U})$ and $\boldsymbol{\alpha}(\mathcal{U}) = \bigcap_{k \leq 0} \bigcup_{n \leq k} \mathcal{F}^n(\mathcal{U})$, cf. [18].) As before attractors and repeller form (finite) distributive lattices denoted as $\mathsf{Att}(\mathcal{X}, \mathcal{F})$ and $\mathsf{Rep}(\mathcal{X}, \mathcal{F})$, respectively. A subset $\mathcal{M} \subset \mathcal{X}$ is called a Morse set if $\mathcal{M} = \mathcal{A} \cap \mathcal{R}$ for some attractor $\mathcal{A}$ and some repeller $\mathcal{R}$. Even though attractors and repellers are not necessarily invariant, Morse sets are. A poset $(\mathsf{M}, \leq)$ consisting of non-empty, pairwise disjoint, invariant subsets $\mathcal{M} \subset \mathcal{X}$ is called a *Morse representation* for $(\mathcal{X}, \mathcal{F})$ if for every complete orbit $\{\xi_n\}$ such that $\xi_{n'} \in \mathcal{M}'$ and $\xi_{n''} \in \mathcal{M}''$ for some $n' < n''$, then $\mathcal{M}' < \mathcal{M}''$. The theory of Morse representations and distributive lattices of attractors remains unchanged for combinatorial systems. The combinatorial theory can best be summarized as a generalized version of Birkhoff's theorem for finite digraphs, cf. [13, 14].

As before attractors and repellers may be defined via larger sets in the digraph. Forward invariant sets $\mathcal{U}$ satisfy the property $\mathcal{F}(\mathcal{U}) \subset \mathcal{U}$ and form the lattice
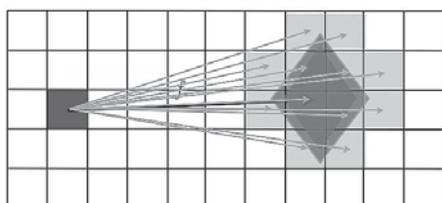
Figure 7 An example of a grid and a multivalued map $\mathcal{F}$; for any $\xi \in \mathcal{X}$, $\mathcal{F}(|\xi|) = \{\eta \in \mathcal{X} \mid B_\varepsilon(f(|\xi|)) \cap \eta \mid \neq \varnothing\}$, cf. [1].
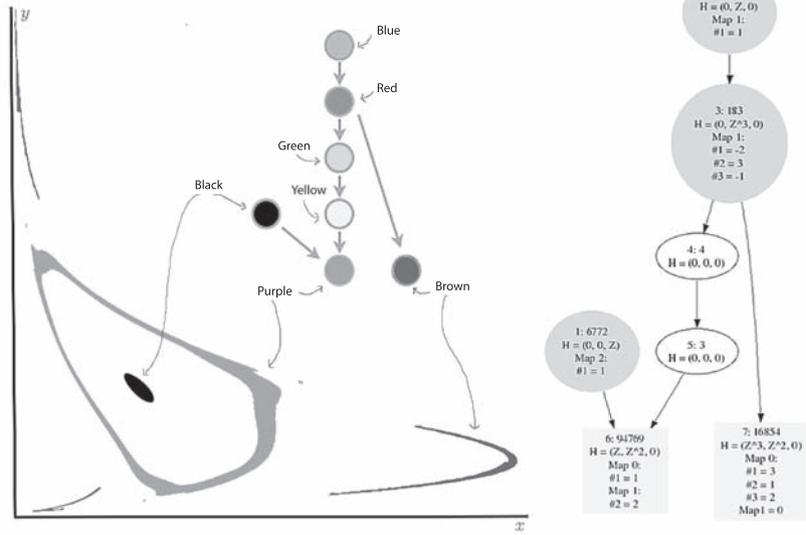
Figure 8 The strongly connected components $\mathsf{SC}(\mathcal{X}, \mathcal{F})$ for $\mathcal{F}$ realized in the plane (left), the reduced graph $\mathsf{RC}(\mathcal{X}, \mathcal{F})$ with cyclic strongly connected components for $\mathcal{F}$ (middle) and the Conley index computations (right), cf. [1].

$\mathsf{Invset}^+(\mathcal{X}, \mathcal{F})$. Similarly, backward invariant sets form the lattice $\mathsf{Invset}^-(\mathcal{X}, \mathcal{F})$. This yields the analogue of Diagram (2):

$$
\begin{array}{ccc}
\mathsf{Invset}^+(\mathcal{X}, \mathcal{F}) & \xleftrightarrow{\; c \;} & \mathsf{Invset}^-(\mathcal{X}, \mathcal{F}) \\
{\scriptstyle \omega} \downarrow & & \downarrow {\scriptstyle \alpha} \\
\mathsf{Att}(\mathcal{X}, \mathcal{F}) & \xleftrightarrow{\; * \;} & \mathsf{Rep}(\mathcal{X}, \mathcal{F})
\end{array} \quad (7)
$$

Applying Birkhoff's representation theorem yields representations for $\mathsf{J}(\mathsf{Invset}^+(X, \mathcal{F}))$ and $\mathsf{J}(\mathsf{Att}(X, \mathcal{F}))$ given by the posets $\mathsf{SC}(X, \mathcal{F})$ and $\mathsf{RC}(X, \mathcal{F})$ respectively. The elements in $\mathsf{SC}(X, \mathcal{F})$ are obtained by intersecting forward and backward invariant sets as described before, and the elements in $\mathsf{RC}(X, \mathcal{F})$ are Morse sets in $\mathcal{F}$ which are obtained by intersecting attractors and repellers. In the language of digraph theory the poset $\mathsf{SC}(X, \mathcal{F})$ is the (acyclic) digraph of strongly connected components and the poset $\mathsf{RC}(X, \mathcal{F})$ is the subgraph of cyclic strongly connected components. (A digraph is *strongly connected* if there exists a path between each ordered pair of vertices. A *strongly connected component* of a digraph is maximal strongly connected subgraph. The *cyclic strongly connected components*, or *recurrent components* are the strongly connected components that either consist of a single vertex with a self-connection, or consist of multiple points.) From the functoriality we have the canonical order inclusion $\mathsf{RC}(X, \mathcal{F}) \hookrightarrow \mathsf{SC}(X, \mathcal{F})$. The acyclic strongly connected components in $\mathsf{SC}(X, \mathcal{F}) \backslash \mathsf{RC}(X, \mathcal{F})$ consists of

one grid element ξ and for such elements $\mathrm{Inv}(|\xi|, f) = \varnothing$. For the components in $\mathsf{RC}(X, \mathcal{F})$ this is not necessarily true and the maximal invariant inside such a component may not be empty. Tarjan's algorithm provides an efficient linear time algorithms in the number of vertices that computes the strongly connected components of a digraph and thus a finest Morse representation, see cf. [5]. The idea of discretizing space and combinatorializing dynamics is best explained by an example.

The following model is used in population dynamics to describe the interaction of two species x and y. The dynamics is given by the mapping

$$
f(x, y; \mu_1, \ldots, \mu_4) = \begin{pmatrix} (\mu_1 + \mu_2) e^{-\mu_3(x+y)} \\ \mu_4 x \end{pmatrix}, \quad (8)
$$

where $\mu_1, \ldots, \mu_4$ are positive parameters in the model, see [25, 26]. Choose the parameters to be $\mu_1 = 22.5$, $\mu_2 = 25$, $\mu_3 = 0.1$ and $\mu_4 = 0.7$. Observe that the square $X = \{(x, y) \mid 0 \leq x \leq a, 0 \leq y \leq b\}$, with $a \geq \mu_1 + \mu_2$ and $b \geq \mu_4(\mu_1 + \mu_2)$, is forward invariant with respect to f and therefore consider $f : X \to X$.

We choose a rectangular grid on X with elements of size $\mathrm{diam}(|\xi|) \leq d$ and construct a multivalued mapping $\mathcal{F}$ that maps grid elements to sets of grid elements, cf Figure 7. In Example 9 we choose

$$
\mathcal{F}(\xi) = \{\eta \in \mathcal{X} \mid B_\varepsilon(f(|\xi|)) \cap \eta \neq \varnothing\}, \quad (9)
$$

factoring in a numerical error ε, cf. Figures 7 and 8, cf. [1].

Computability and interpretation

The dynamics captured by an outer approximation $\mathcal{F}$ has a direct interpretation to the actual dynamics of (X, f). As we indicated before the multivalued mapping $\mathcal{F}$ may be interpreted as a digraph. Figure 8 gives a realization of the strongly connected components of the digraph, color coded (left). By identifying the cyclic strongly connected components with vertices, a schematic picture of the dynamics emerges via $\mathrm{RC}(\mathcal{X}, \mathcal{F})$ in terms of the digraph Figure 8 (middle).

Realization

Define the *evaluation mapping* $|\cdot| : \mathsf{Set}(\mathcal{X}) \to \mathrm{RG}(X)$ via $\mathcal{U} \mapsto \bigcup_{\xi \in \mathcal{U}} |\xi|$. A trapping region N is called an *attracting block* if $f(N) \subset \mathrm{int}(N)$. Attracting blocks that are also regular and closed form the lattice denoted by $\mathrm{ABlock}_{\mathrm{RG}}(X, f) \subset \mathrm{RG}(X)$. If $\mathcal{U} \in \mathrm{Invset}^+(\mathcal{X}, \mathcal{F})$ is a lattice homomorphism. As a matter of fact the following commuting diagram exists, cf. [16]:

$$
\begin{array}{ccc}
\mathrm{Invset}^+(\mathcal{X}, \mathcal{F}) & \xrightarrow{\;|\cdot|\;} & \mathrm{ABlock}_{\mathrm{RG}}(X, f) \\
{\scriptstyle \omega}\downarrow & & \downarrow{\scriptstyle \omega} \quad (10) \\
\mathrm{Att}(\mathcal{X}, \mathcal{F}) & \xrightarrow{\;\omega(|\cdot|)\;} & \mathrm{Att}(X, f)
\end{array}
$$

The dual diagram for backward invariant sets, repelling blocks and repellers is obtained via the duality mapping

$$
\#: \mathrm{ABlock}_{\mathrm{RG}}(X, f) \longleftrightarrow \mathrm{RBlock}_{\mathrm{RG}}(X, f),
$$

where $N^\# = \mathrm{cl}(N^c)$, the 'complement' in $\mathrm{RG}(X)$. Let $\mathsf{N}_{\mathcal{F}}$ and $\mathsf{A}_{\mathcal{F}}$ be the images of $\mathrm{Invset}^+(\mathcal{X}, \mathcal{F})$ and $\mathrm{Att}(\mathcal{X}, \mathcal{F})$ under the mappings $|\cdot|$ and $\omega(|\cdot|)$, respectively. This yields the commuting diagram

$$
\begin{array}{ccc}
\mathrm{Invset}^+(\mathcal{X}, \mathcal{F}) & \xleftarrow{\;|\cdot|\;} & \mathsf{N}_{\mathcal{F}} \\
{\scriptstyle \omega}\downarrow & & \downarrow{\scriptstyle \omega} \quad (11) \\
\mathrm{Att}(\mathcal{X}, \mathcal{F}) & \xrightarrow{\;\omega(|\cdot|)\;} & \mathsf{A}_{\mathcal{F}}
\end{array}
$$

By Birkhoff duality and functoriality we obtain a commuting diagram linking the strongly connected components in $\mathcal{F}$ to tesselated Morse decompositions:

$$
\begin{array}{ccc}
\mathrm{SC}(\mathcal{X}, \mathcal{F}) & \xleftarrow{\;|\cdot|\;} & \mathsf{T}(\mathsf{N}_{\mathcal{F}}) \\
\uparrow & & \uparrow \quad (12) \\
\mathrm{RC}(\mathcal{X}, \mathcal{F}) & \xleftarrow{\;\pi\;} & \mathrm{M}(\mathsf{A}_{\mathcal{F}})
\end{array}
$$

We conclude that $\mathrm{RC}(\mathcal{X}, \mathcal{F})$ yields a tesselated Morse decomposition

$$
\pi : \mathrm{M}(\mathsf{A}_{\mathcal{F}}) \mapsto \mathrm{RC}(\mathcal{X}, \mathcal{F}) \\
\mapsto \mathrm{SC}(\mathcal{X}, \mathcal{F}) \longleftrightarrow \mathsf{T}(\mathsf{N}_{\mathcal{F}}).
$$

The mapping $U \mapsto \mathrm{Inv}(U, f)$ acts as a left inverse. In particular, $\mathrm{M}(\mathsf{A}_{\mathcal{F}}) = \{\mathrm{Inv}(|\mathcal{M}|, f) \neq \varnothing \mid \mathcal{M} \in \mathrm{RC}(\mathcal{X}, \mathcal{F})\}$ and $\mathsf{T}(\mathsf{N}_{\mathcal{F}}) = \{|\mathcal{U}| \mid \mathcal{U} \in \mathrm{SC}(\mathcal{X}, \mathcal{F})\}$. From the theory of Morse decompositions as described before we have that the composition $\pi : \mathrm{M}(\mathsf{A}_{\mathcal{F}}) \to \mathsf{T}(\mathsf{N}_{\mathcal{F}})$ is a Morse decomposition.

What does this approach tell us about the example given by (8)? The middle graph $\mathrm{RC}(\mathcal{X}, \mathcal{F})$ in Figure 8 gives a candidate for a Morse representation. However, it is not straightforward to determine whether the evaluated Morse sets $|\mathcal{M}|$ contain actual Morse sets for f. A sophisticated topological tool called the Conley index provides sufficient conditions which imply that $\mathrm{Inv}(|\mathcal{M}|, f) \neq \varnothing$, cf. Figure 8 (right). By construction a Morse set $\mathcal{M}$ is given by $\mathcal{M} = \mathcal{U} \setminus \overleftarrow{\mathcal{U}}$, where $\mathcal{U} \in \mathsf{J}(\mathrm{Att}(\mathcal{X}, \mathcal{F}))$ and $\overleftarrow{\mathcal{U}}$ is the unique predecessor. The version of the Conley index used in Figure 8 is the set of all non-zero eigenvalues of

$$
\widehat{f}_* : H_*(N/\overleftarrow{N}, [\overleftarrow{N}]) \to H_*(N/\overleftarrow{N}, [\overleftarrow{N}]),
$$

restricted to the torsion free part of $H_*(N/\overleftarrow{N}, [\overleftarrow{N}])$, where $N = |\mathcal{U}|$, $\overleftarrow{N} = |\overleftarrow{\mathcal{U}}|$, and

$$
\widehat{f}([x]) = \begin{cases} [f(x)] & \text{if } x, f(x) \in N \setminus \overleftarrow{N}, \\ [\overleftarrow{N}] & \text{otherwise,} \end{cases}
$$

is continuous by construction. The above arguments provide a second reason why the algebra of distributive lattices plays a crucial role in determining non-trivial Morse sets.

If we employ the Conley index we conclude for the model given in that there exist at least two attractors, two non-trivial (saddle-like) Morse sets and one repeller. The green and yellow vertices in Figure 8 (middle) may contain no invariant dynamics. The poset $\mathrm{RC}(\mathcal{X}, \mathcal{F})$ combinatorializes the dynamics between the Morse sets. The draw back is that the elements $|\mathcal{M}|$, $\mathcal{M} \in \mathrm{RC}(\mathcal{X}, \mathcal{F})$, do not tile the space X in general. The elements in $\mathrm{SC}(\mathcal{X}, \mathcal{F})$ do, but the number of elements $\{\xi\} \in \mathrm{SC}(\mathcal{X}, \mathcal{F}) \setminus \mathrm{RC}(\mathcal{X}, \mathcal{F})$ is many orders of magnitude larger than $\mathrm{RC}(\mathcal{X}, \mathcal{F})$. If we can construct a sublattice $\mathsf{U} \subset \mathrm{Invset}^+(\mathcal{X}, \mathcal{F})$ which is isomorphic to $\mathrm{Att}(\mathcal{X}, \mathcal{F})$ then the sets $|\mathcal{U} \setminus \overleftarrow{\mathcal{U}}|$, with

$\mathcal{U} \in \mathsf{J}(\mathsf{U})$, tile the space X. This task is much harder and is related to the question of realizability and convergence: can every structure be realized provided the grid is sufficiently fine? This question is addressed in the next subsection.

Convergence

By refining a grid — subdividing, or choosing a smaller grid-size — more detailed information about the dynamics of (X, f) may be obtained. Keeping in mind the example in let $(\mathcal{X}_n, \mathcal{F}_n)$ be a sequence of refinements by subdividing the grid elements and by choosing the sequence of multivalued mappings accordingly, cf. (9). In [16] it was proved that for a refining sequence of outer approximations $\mathcal{F}_n : \mathcal{X}_n \rightrightarrows \mathcal{X}_n$ and any finite sublattice $\mathsf{A} \subset \mathrm{Att}(X, f)$, there exists an $n_\mathsf{A} > 0$ such that A lifts into $\mathrm{Invset}^+(\mathcal{X}_n, \mathcal{F}_n)$ for all $n \geq n_\mathsf{A}$, i.e. there exist embeddings $\mathsf{A} \hookrightarrow \mathrm{Invset}^+(\mathcal{X}_n, \mathcal{F}_n)$ such that $\omega(|\mathsf{U}|) = \mathsf{A}$, where $\mathsf{U} \subset \mathrm{Invset}^+(\mathcal{X}_n, \mathcal{F}_n)$ is the isomorphic image of A. This convergence result has consequences for the existence of Morse tilings for given a Morse representation. Let $\mathcal{F} = \mathcal{F}_0$ and consider $\mathsf{A}_{\mathcal{F}}$ and the associated Morse representation $\mathrm{M}(\mathsf{A}_{\mathcal{F}})$. Let U_n be the image of $\mathsf{A}_{\mathcal{F}} \hookrightarrow \mathrm{Invset}(\mathcal{X}_n, \mathcal{F}_n)$, $n \geq n_{\mathsf{A}_{\mathcal{F}}}$. Then, the isomorphism

$$
\omega : \mathsf{N}_n = |\mathsf{U}_n| \longleftrightarrow \mathsf{A}_{\mathcal{F}},
$$

yields a tesselated Morse decomposition $\mathrm{M}(\mathsf{A}_{\mathcal{F}}) \longleftrightarrow \mathsf{T}(\mathsf{N}_n)$. From the point of view of computation this result does not provide practical ways to find Morse tilings and tesselated Morse decompositions. A more practical method may be derived as follows.

The Morse decomposition $\pi : \mathrm{M}(\mathsf{A}_{\mathcal{F}}) \mapsto \mathrm{RC}(\mathcal{X}, \mathcal{F})$ is the starting point, which yields the tesselated Morse decomposition $\mathrm{M}(\mathsf{A}_{\mathcal{F}}) \mapsto \mathsf{T}(\mathsf{N}_{\mathcal{F}}) \cong \mathrm{SC}(\mathcal{X}, \mathcal{F})$. The objective is to coarsen the poset $\mathrm{SC}(\mathcal{X}, \mathcal{F})$ such we obtain a Morse tiling T which is isomorphic to $\mathrm{RC}(\mathcal{X}, \mathcal{F})$. From Birkhoff duality it follows that the existence of a lift k in Diagram (13),

$$
\begin{array}{ccc}
\mathrm{O}(\mathrm{SC}(\mathcal{X}, \mathcal{F})) & \longleftrightarrow & \mathrm{Invset}^+(\mathcal{X}, \mathcal{F}) \\
\downarrow & {\scriptstyle k} \nearrow & \downarrow{\scriptstyle \omega} \quad (13) \\
\mathrm{O}(\mathrm{RC}(\mathcal{X}, \mathcal{F})) & \longleftrightarrow & \mathrm{Att}(\mathcal{X}, \mathcal{F})
\end{array}
$$

is equivalent to the existence of an order-

surjection

$$j : \mathrm{SC}(\mathcal{X}, \mathcal{F}) \twoheadrightarrow \mathrm{RC}(\mathcal{X}, \mathcal{F}), \qquad (14)$$

with $j \circ i = \mathrm{id}$ on $\mathrm{RC}(\mathcal{X}, \mathcal{F})$. If the grid size is not small enough then such an order-surjection j need not exist! However, convergence helps out in this case. Let $\mathcal{X}_n$ be a sequence of refining grids with $\mathrm{diam}(\mathcal{X}_n) \to 0$ and let $\mathcal{F}_n : \mathcal{X}_n \rightrightarrows \mathcal{X}_n$ be a sequence of outer approximations as given in with $\mathcal{F} = \mathcal{F}_0$. Then, there exists an $n_{\mathrm{A}_\mathcal{F}} > 0$ such that $\mathrm{RC}(\mathcal{X}, \mathcal{F})$ allows order-surjections $j_n : \mathrm{SC}(\mathcal{X}_n, \mathcal{F}_n) \twoheadrightarrow \mathrm{RC}(\mathcal{X}, \mathcal{F})$ for all $n \geq n_{\mathrm{A}_\mathcal{F}}$, cf. [13]. The mappings j_n provide the rules for coarsening $\mathrm{SC}(\mathcal{X}_n, \mathcal{F}_n)$ and we obtain tesselated Morse decompositions

$$\pi : \mathrm{M}(\mathrm{A}_\mathcal{F}) \rightarrowtail \mathrm{RC}(\mathcal{X}, \mathcal{F}) \hookleftarrow \mathrm{T}(\mathrm{N}_n), \qquad (15)$$

where $\mathrm{N}_n = |\, k_n(\mathrm{RC}(\mathcal{X}, \mathcal{F})) \,|$ and

$$k_n : \mathrm{O}(\mathrm{RC}(\mathcal{X}, \mathcal{F})) \rightarrowtail \mathrm{Invset}(\mathcal{X}_n, \mathcal{F}_n).$$

We do not expatiate on methods of computing j_n. Given j_n, for n sufficiently large, provides a complete combinatorial picture of the dynamics of (X, f) within a given resolution: every point $x \in X$ is contained in a tile $T \in \mathrm{T}(\mathrm{N}_n)$ and the order relation on $\mathrm{T}(\mathrm{N}_n)$ provides the possible Morse sets to which the dynamics converges as $t \to \pm\infty$. The graph $\mathrm{RC}(\mathcal{X}, \mathcal{F})$ in Figure 8 (left) now provides a complete description of the dynamics within the chosen resolution since in the tesselated Morse decomposition $\mathrm{M}(\mathrm{A}_\mathcal{F}) \rightarrowtail \mathrm{T}(\mathrm{N}_n)$, the poset $\mathrm{T}(\mathrm{N}_n)$ is a Morse tiling. The above arguments provide a third reason why the algebraic approach provides the appropriate framework for computational combinatorializations of the dynamics.

The relation between parameter dependence in systems and combinatorial systems is still unclear. Since the way of encoding dynamical information into a combinatorial system is insensitive under small perturbations the question arises how the algebraic structures behave under large parameter variations — continuations. A good understanding of the algebra involved may give algebraic criteria for detecting certain types of global bifurcations. The ideas in [1] about studying dynamical systems with a number of parameters could be comprised in a database with information on for instance Morse decompositions for a finite number of parameter values. With the appropriate algebra at hand various properties of the system can be found in combination with variation in parameters. We believe that the answer should come from the algebra of finite distributive lattices and their morphisms. This requires a formulation of local continuation of Morse representations in terms sheafs of Morse representations and attractor lattices.   ⬫⋯

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