Problemen

Problem Section

This Problem Section is open to everyone; everybody is encouraged to send in solutions and propose problems. Group contributions are welcome.
For each problem, the most elegant correct solution will be rewarded with a book token worth €20. At times there will be a Star Problem, to which the proposer does not know any solution. For the first correct solution sent in within one year there is a prize of €100.
When proposing a problem, please either include a complete solution or indicate that it is intended as a Star Problem. Electronic submissions of problems and solutions are preferred (problems@nieuwarchief.nl).
The deadline for solutions to the problems in this edition is 1 September 2013.

Problem A (proposed by Gerard Renardel de Lavalette)

We have two hourglasses, A for a seconds and B for b seconds, where a and b are relatively prime integers and $0 < a < b$. Let t_0 be an integer with $t_0 \geq b + (\frac{1}{2}a - 1)^2$. Show that A and B can be used to identify the time $t = t_0$ if the upper bulbs are empty at $t = 0$.

Problem B (folklore)

In a two-player game, players take turns drawing a number of coins from a pile that starts with n coins. The first player takes at least one coin from the pile, but not all. In the subsequent turns, each player takes at least one coin, and at most twice the number of coins taken in the previous turn. The player who takes the last coin wins.
For which numbers n can the first player win?

Problem C (proposed by Bas Edixhoven and Maarten Derickx)

Let $ABCD$ be a convex quadrilateral inside a plane U in $\mathbb{R}^3$. Suppose that $ABCD$ is not a parallelogram. Show that there exist a plane V in $\mathbb{R}^3$ and a point $P \in \mathbb{R}^3 - (U \cup V)$ such that if a light source is placed in P, then the shadow of $ABCD$ on V is a square.

Edition 2012-4 We received solutions from Yagub Aliyev and Dursun Çalişkan (Baku), Charles Delorme (Paris), Pieter de Groen (Brussels), Alex Heinis (Hoofddorp), Thijmen Krebs (Nootdorp), John Simons (Groningen), Traian Viteam (Punta Arenas) and Robert van der Waall (Huizen).

Problem 2012-4/A Let $\varphi(n)$ denote the Euler totient function. Find the set of limit points of the sequence $(\varphi(n)/n)_{n=1}^{\infty}$.

Solution We received solutions from Yagub Aliyev and Dursun Çalişkan, Charles Delorme, Pieter de Groen, Alex Heinis, Thijmen Krebs, Traian Viteam and Robert van der Waall. The following solution is based on that of Thijmen Krebs. The book token goes to Thijmen Krebs.
We start by noting that $\varphi(n) = n\prod_{p|n}(1 - \frac{1}{p})$ for all positive integers n.
Note that the set L of limit points of the sequence must lie inside the closed interval $[0, 1] \subseteq \mathbb{R}$, as the sequence itself lies inside $[0, 1]$. We show that $L = [0, 1]$.
Let $P = \{p_1, p_2, \ldots\}$ (with $p_1 < p_2 < \cdots$) denote the set of primes. Then the subsequence $(\varphi(p_n)/p_n)_{n=1}^{\infty} = \left(\frac{p_n - 1}{p_n}\right)_{n=1}^{\infty}$ has limit 1, so $1 \in L$.
Now let $x \in [0, 1)$. To show that $x \in L$, it suffices to show that for all $\epsilon > 0$, there exists a positive integer n such that $|x - \varphi(n)/n| < \epsilon$. We assume without loss of generality that $x + \epsilon < 1$. Hence there exists a $p_s \in P$ with $x + \epsilon < 1 - \frac{1}{p_s}$ and $\epsilon > \frac{1}{p_s - 1}$. As $\prod_{i=s}^{\infty}(1 - \frac{1}{p_i}) = 0$, it follows that there is an integer $t \geq s$ such that

$$\prod_{i=s}^{t}(1 - \tfrac{1}{p_i}) \geq x + \tfrac{1}{p_s - 1} > \prod_{i=s}^{t+1}(1 - \tfrac{1}{p_i}) =: \alpha,$$

Problemenrubriek NAW
Mathematisch Instituut
Universiteit Leiden
Postbus 9512
2300 RA Leiden
problems@nieuwarchief.nl
www.nieuwarchief.nl/problems

hence

$$\begin{aligned} x+\epsilon > x+\tfrac{1}{p_s-1} &> \alpha \geq \left(x+\tfrac{1}{p_s-1}\right)\left(1-\tfrac{1}{p_{t+1}}\right) \\ &> \left(x+\tfrac{1}{p_s-1}\right)\left(1-\tfrac{1}{p_s}\right) = x+\tfrac{1}{p_s}(1-x) > x. \end{aligned}$$

This completes the proof as $\alpha = \varphi(n)/n$, where $n = \prod_{i=s}^{t+1} p_i$. ☐

Remark Thanks to Robert van der Waall for pointing out a reference for this problem, namely: A. Schinzel, W. Sierpinski, *Bull. Acad. Polon. Sci.*, Cl. III, Vol. 2 (1954), pp 463–466, and Vol. 3 (1955), pp. 415–419.

Problem 2012-4/B Find nonzero integers c_0, c_1, c_2, c_3 such that the sequence given by $a_1 = 1$, $a_2 = 12$, $a_3 = 68$, $a_4 = 504$ and

$$a_{n+4} = c_0 a_n + c_1 a_{n+1} + c_2 a_{n+2} + c_3 a_{n+3} \quad (n > 0)$$

consists of positive terms and has the property that a_m divides a_n whenever m divides n.

Solution We received solutions from Charles Delorme, Alex Heinis, Thijmen Krebs and John Simons. The solution below is based on the solution of Thijmen Krebs. The book token goes to Alex Heinis.
Thanks to Frits Beukers for providing inspiration for this problem.
Let $(F_n)_{n\geq 1}$ be the Fibonacci sequence, which is the sequence defined by $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ $(n > 0)$. Let $\varphi = \frac{1}{2}\left(1+\sqrt{5}\right)$ be the positive root of $X^2 - X - 1 = 0$ and let $\psi = 1 - \varphi$ be the other one. Then $F_n = (\varphi^n - \psi^n)/\sqrt{5}$. The Fibonacci sequence is a divisibility sequence: if m divides n, say $n = km$, then F_m divides F_n because the quotient

$$\frac{F_n}{F_m} = \frac{\varphi^{km} - \psi^{km}}{\varphi^m - \psi^m} = \varphi^{(k-1)m} + \varphi^{(k-2)m}\psi^m + \cdots + \psi^{(k-1)m}$$

is an algebraic integer (because φ and ψ are, as they are the roots of the monic polynomial $X^2 - X - 1$) that is a rational number. Therefore, it is an integer, i.e., F_m divides F_n. We are ready to state our claim: the sequence

$$a_n = \frac{F_{3n}F_{2n}}{2F_n} = \frac{1}{2\sqrt{5}}\left(\varphi^{3n} - \psi^{3n}\right)\left(\varphi^n + \psi^n\right)$$

has positive terms, has the correct initial values, and satisfies the divisibility condition.

- Each a_n is a positive integer. Indeed, $a_n > 0$ is clear, and we know (by the divisibility property of Fibonacci proven above) that $2 = F_3$ divides F_{3n} and F_n divides F_{2n}. The initial values are $a_1 = 1$, $a_2 = 12$, $a_3 = 68$ and $a_4 = 504$.
- If k is odd, then a_n divides a_{kn}. Indeed, we already know that $F_{3n}/2$ divides $F_{3kn}/2$ and we can compute that F_{2n}/F_n divides F_{2kn}/F_{kn}:

  $$\frac{F_{2kn}/F_{kn}}{F_{2n}/F_n} = \frac{\varphi^{kn} + \psi^{kn}}{\varphi^n + \psi^n} = \sum_{i=0}^{k-1} (-1)^i \varphi^{(k-1-i)n}\psi^{in}$$

  is a rational algebraic integer, hence an integer.
- If k is even, then a_n divides a_{kn}. Indeed, k being even implies $3kn$ being divisible by $6n$. So F_{6n} divides F_{3kn} and F_{kn} divides F_{2kn}, so by virtue of

  $$\frac{a_{kn}}{a_n} = \frac{F_{3kn}F_{2kn}/F_{kn}}{F_{3n}F_{2n}/F_n} = \frac{F_{3kn}}{F_{6n}} \cdot \frac{F_{2kn}}{F_{kn}} \cdot \frac{F_{6n}F_n}{F_{3n}F_{2n}}$$

  it suffices to show that $F_{3n}F_{2n}$ divides $F_{6n}F_n$. To prove this, one notes that their quotient equals $\varphi^{2n} + \varphi^n\psi^n + \psi^{2n}$, which is a rational integer.

This concludes the proof of the claim.

It remains to find a fourth order linear recurrence relation for this sequence. We do this by computing that

$$a_n = \frac{1}{2\sqrt{5}}\left(\varphi^{3n} - \psi^{3n}\right)(\varphi^n + \psi^n) = \frac{1}{2\sqrt{5}}\left(\left(\varphi^4\right)^n + \left(\varphi^3\psi\right)^n - \left(\varphi\psi^3\right)^n - \left(\psi^4\right)^n\right),$$

and that

$$(X-\varphi^4)(X-\varphi^3\psi)(X-\varphi\psi^3)(X-\psi^4) = X^4 - 4X^3 - 19X^2 - 4X + 1.$$

These two equations imply that

$$a_{n+4} = 4a_{n+3} + 19a_{n+2} + 4a_{n+1} - a_n.$$

We conclude that $c_0 = -1$, $c_1 = 4$, $c_2 = 19$, $c_3 = 4$ satisfy the conditions.

Remark Alternatively, one could try to find a sequence of the form

$$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \cdot \frac{\gamma^n - \delta^n}{\gamma - \delta}.$$

One deduces from the values of a_2, a_3 and a_4 that $\alpha\gamma, \alpha\delta, \beta\gamma, \beta\delta$ are the roots of the polynomial

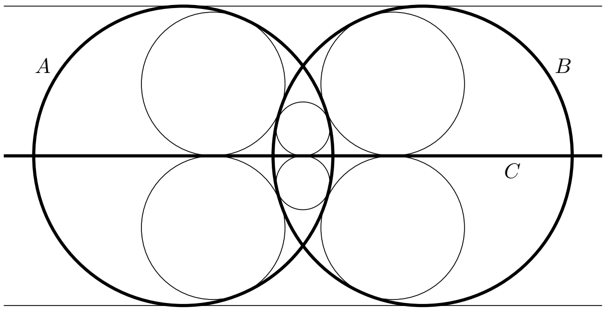
$$(X-\alpha\gamma)(X-\alpha\delta)(X-\beta\gamma)(X-\beta\gamma) = X^4 - 12X^3 + 51X^2 - 300X + 625.$$

So we could guess that $c_0 = -625$, $c_1 = 300$, $c_2 = -51$ and $c_3 = 12$ satisfy the conditions. They do, but the proof of this statement is slightly more technical than the proof above. (Although the main ideas coincide.)

Problem 2012-4/C A circle in $\mathbb{R}^2$ is called *Apollonian* if its centre coordinates and radius are all integers. Do there exist eleven distinct Apollonian circles $A, B, C, T_1, \dots, T_8$ such that for $i = 1, \dots, 8$, the circle T_i is tangent to A, B, and C?

Solution We received solutions from Charles Delorme, Alex Heinis and Thijmen Krebs. The solution below is based on all of their solutions. The book token goes to Charles Delorme. The answer to the problem as stated is trivially affirmative, as the circles C_r with centre $(r, 0)$ and radius r (for $r \in \mathbb{Z}$) are all tangent to each other at the origin. We therefore added the originally-intended extra requirement that the centres of the circles A, B and C not be collinear in the previous (March) issue. We will show that this still allows many examples. It suffices to show the existence of rational, rather than integral, solutions, as we may multiply by a common denominator in the end.

One way to find examples is to first look for degenerate cases, where some of the circles are lines; then inversion in any circle with rational radius and rational centre P not lying on any of the lines and circles, yields a non-degenerate example as desired, provided that P has rational distance to all the lines. For example, take the configuration depicted below, where the circles A and B have centres $(\pm b, 0)$ and radius r, the (degenerate) circle C is given by $y = 0$, while two



of the remaining eight circles have centres $(0, \pm c)$, four have centres $(\pm a, \pm d)$, and the last two (degenerate) circles are given by $y = \pm r$ for appropriate a, b, c, d, r. Assuming $r > b > a > 0$, it is not hard to find that the tangency conditions are equivalent to the equalities $a^2 + b^2 = r^2$ and $rd = ab$ and $2rc = a^2$. The shown figure is for the Pythagorean triple $(a, b, r) = (3, 4, 5)$, which determines $c = 9/10$ and $d = 12/5$.

After inversion, this yields a configuration with eleven circles as required, with several points where three circles are tangent to each other. There are also configurations where no three circles are concurrent. One example is depicted below, with the centre $(x, y) = (a/d, b/d)$ and the radius $r = c/d$ for each circle as in the following table.

	a	b	c	d
A	0	0	1	1
B	560	0	1649	1551
C	0	420	949	851
T_1	40	30	7	43
T_2	−40	−30	7	57
T_3	1200	−49	210	991
T_4	−1200	49	210	1411
T_5	−49	1200	280	921
T_6	49	−1200	280	1481
T_7	21	28	120	85
T_8	21	28	120	155

To find this example, the equations expressing tangency were simplified by the additional requirements that the centres P, Q and R of the circles A, B and C form a right angle at P, that the circles A and B intersect on the line PR, and that the circles A and C intersect on the line PQ. We leave it to the reader to verify that this implies that the entire picture is invariant under the composition of inversion with respect to A and reflection in P. Hence, the circles $T_1, \ldots, T_8$ split up into four pairs, where each circle in a pair can be obtained from the other by a homothety with respect to P. Since P is chosen to be the origin, this is clearly reflected in the table.