This Problem Section is open to everyone; everybody is encouraged to send in solutions and propose problems. Group contributions are welcome.

For each problem, the most elegant correct solution will be rewarded with a book token worth 20 Euro. At times there will be a Star Problem, to which the proposer does not know any solution. For the first correct solution sent in within one year there is a prize of 100 Euro.

When proposing a problem, please either include a complete solution or indicate that it is intended as a Star Problem. Electronic submissions of problems and solutions are preferred (problems@nieuwarchief.nl).

The deadline for solutions to the problems in this edition is 1 March 2013.

**Problem A** (proposed by Jan Turk)

Let $\phi(n)$ denote the Euler totient function. Find the set of limit points of the sequence $\left( \frac{\phi(n)}{n} \right)_{n=1}^{\infty}$.

**Problem B**

Find nonzero integers $c_0, c_1, c_2, c_3$ such that the sequence given by

\[ a_1 = 1, \quad a_2 = 12, \quad a_3 = 68, \quad a_4 = 504, \quad a_{n+4} = c_0 a_n + c_1 a_{n+1} + c_2 a_{n+2} + c_3 a_{n+3} \quad (n > 0) \]

consists of positive terms and has the property that $a_m$ divides $a_n$ whenever $m$ divides $n$.

**Problem C** (proposed by Johanna Winterink)

A circle in $\mathbb{R}^2$ is called *Apollonian* if its centre coordinates and radius are all integers.

Do there exist eleven distinct Apollonian circles $A, B, C, T_1, \ldots, T_8$ such that for $i = 1, \ldots, 8$, the circle $T_i$ is tangent to $A, B,$ and $C$?

**Edition 2012-2** We received solutions from Wouter Cames van Batenburg (Leiden), Cor Hurkens (Eindhoven), Thijmen Krebs (Nootdorp), José H. Nieto (Maracaibo) and Hans Zantema (Eindhoven).

**Problem 2012-2/A** Let $P$ and $Q$ be distinct points in the plane. Let $n \geq 2$. Assume $n$ distinct lines through $P$ but not through $Q$ are given, as well as $n$ distinct lines through $Q$ but not through $P$. Let $T$ be a collection of $2n$ intersection points of these lines. Suppose that the (unoriented) angle between the lines $RP$ and $RQ$ is the same for all $R \in T$, and not a multiple of $\frac{1}{4}\pi$. Show that $T$ can be partitioned into subsets of at least three elements each, such that every subset consists of the vertices of a regular polygon.

**Rectification.** The common angle in this problem should not be a multiple of $\pi/4$. (Thanks to Thijmen Krebs for pointing this out.)

**Solution** We received a correct solution from Thijmen Krebs.

All angles are oriented angles modulo $\pi$, unless stated otherwise. Let $\alpha$ be the unoriented angle modulo $\pi$ of the common angle of the $\angle PRQ$, where $R \in T$.

**Observation 1.** Every line through $P$ (resp. $Q$) contains exactly two points of $T$.

**Proof.** Let $L$ be a line through $P$. As $Q$ is not on this line, there is a unique isosceles triangle with base inside $L$, top $Q$, and base angles $\alpha$. Hence there are at most two points of $T$ on any given line through $P$. But since we have $n$ lines going through $P$, and $\#T = 2n$, it must follow that every line must contain exactly two points of $T$. The same argument holds for $Q$. \[ \square \]

**Observation 2.** The set $T$ is a subset of the union of two distinct circles intersecting at $P$ and $Q$. 

Problem. Note that by the inscribed angle theorem, the subset $T_*$ of $T$ consisting of the points $R \in T$ such that $\angle PRQ = \alpha$ lies on a circle $\Gamma$, containing $P$ and $Q$, and that the subset $T_-$ of $T$ consisting of the points $R \in T$ such that $\angle PRQ = -\alpha$ also lies on a circle $\Gamma$, containing $P$ and $Q$. Moreover, these circles are distinct since $\alpha \neq \frac{\pi}{2}$ by assumption. \hfill \Box

We now define two maps $f_p, f_Q : T_+ \to T_+$ as follows. Let $R \in T_+$. Then $f_p(R)$ (resp. $f_Q(R)$) is the unique intersection point of the line $RP$ (resp. $RQ$) with $\Gamma$, not equal to $P$ (resp. $Q$). This map is well-defined, as for $R \in T_+$, we have $\angle PRf_pRQ = \angle PfrQ = -\alpha$, hence $f_p(R), f_Q(R) \in T_+$ by Observation 4.

Observation 3. The maps $f_p$ and $f_Q$ are bijections. In particular, $\# T_+ = \# T_+ = n$.

Proof. We simply note that the inverse is given by sending $R \in T_+$ to the unique intersection point of the line $RP$ (resp. $RQ$) with $\Gamma$, not equal to $P$ (resp. $Q$). \hfill \Box

Observation 4. The maps $f_p^{-1} f_Q$ and $f_Q f_p^{-1}$ are rotations by $4\alpha$ (as an oriented angle modulo $2\pi$) on $T_+$. and $T_-$, respectively (with centres those of $\Gamma$, and $\Gamma_-$, respectively).

Proof. Let $R \in T_+$. Then $\angle R\Gamma P = \angle R\Gamma f_Q(R)P = \alpha$, it follows that $\angle R\Gamma f_p^{-1} f_Q(R) = \angle R\Gamma f_p(R) = 2\alpha$. Hence if $C_\alpha$ is the centre of $\Gamma$, then $\angle R\Gamma C_\alpha f_p^{-1} f_Q(R) = 4\alpha$, as an oriented angle modulo $2\pi$. The same argument works for $f_Q f_p^{-1}$.

Now we note that the orbits of $T_+$ (resp. $T_-$) under the action of $f_p^{-1} f_Q$ (resp. $f_Q f_p^{-1}$) all have the same length by the above, which hence divides $n$, so it follows that $f_p^{-1} f_Q$ and $f_Q f_p^{-1}$ have order dividing $n$. Hence $4\alpha n = 0$ modulo $2\pi$, so $\alpha = 0$ modulo $\pi/2n$. As we assumed that $\alpha$ was not a multiple of $\frac{\pi}{2}$, it follows that orbits of length at least 2 cannot occur. Orbits of higher length are sets whose vertices form a regular polygon with at least three vertices, so we are done.

Problem 2012-2/B Show that there exist an $n > 1$, a polynomial $P \in \mathbb{Z}[X, Y_1, \ldots, Y_n]$ and an infinite set $S$ of positive integers such that the set

$$\{(y_1, \ldots, y_n) \in \mathbb{Z}^n : P(k, y_1, \ldots, y_n) = 0\}$$

is empty for all $k < 0$ and has precisely $k$ elements for all $k \in S$.

Solution We received a correct solution from Thijmen Krebs.

An example can be deduced from Jacobi’s four-square theorem. It states that for each positive integer $n$, the number of solutions $(y_1, y_2, y_3, y_4) \in \mathbb{Z}^4$ to

$$y_1^2 + y_2^2 + y_3^2 + y_4^2 = p$$

is $r_4(p) = 8 \sum_{d \mid D} d$, where $D$ is the set of divisors of $p$ that are not multiples of 4. In particular, if $p$ is prime we have $r_4(p) = 8(p + 1)$.

Set $n = 4$ and let $P \in \mathbb{Z}[X, Y_1, Y_2, Y_3, Y_4]$ be the polynomial

$$P = 8(Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2) - X.$$

Define $S = \{8(p + 1) : p \text{ prime}\}$. The equation $P(k, y_1, y_2, y_3, y_4) = 0$ has no solutions for $k < 0$. For $k = 8(p + 1) \in S$ the equation reduces to $y_1^2 + y_2^2 + y_3^2 + y_4^2 = p$, which has $r_4(p) = k$ solutions.

Problem 2012-2/C Is it possible to tile a 30 by 30 square grid using the following blocks?

![Blocks](image)

Solution We received correct solutions from Wouter Cames van Batenburg, Cor Hurkens, Thijmen Krebs, José H. Nieto and Hans Zantema. The book contains goes to José H. Nieto.
There exists a tiling as desired. In fact, we can already tile a $10 \times 10$ grid.

Note that we do not even need both types of Z-tiles.
More generally, an $n \times m$ grid can be tiled with the given pieces if and only if $n$ and $m$ are at least 4, $nm$ is divisible by 4, and $(n, m)$ is not $(6, 6)$, $(6, 10)$ or $(10, 6)$. 