Problemen

| Problem Section

This Problem Section is open to everyone; everybody is encouraged to send in solutions and propose problems. Group contributions are welcome.
For each problem, the most elegant correct solution will be rewarded with a book token worth 20 Euro. At times there will be a Star Problem, to which the proposer does not know any solution. For the first correct solution sent in within one year there is a prize of 100 Euro.
When proposing a problem, please either include a complete solution or indicate that it is intended as a Star Problem. Electronic submissions of problems and solutions are preferred (problems@nieuwarchief.nl).
The deadline for solutions to the problems in this edition is 1 December 2012.

Problem A (proposed by Mark Veraar)

Let $(X_n)_{n\geq 1}$ be a sequence of independent random variables with values in $[0,+\infty)$ satisfying

$$\mathbf{P}(X_i > t) = \frac{1}{1+t}$$

for all i and all $t \geq 0$. Let $(c_n)_{n\geq 1}$ be a sequence of positive real numbers. Show that the sequence $(c_n X_n)_{n\geq 1}$ is bounded with probability 1 if and only if the series $\sum_{n=1}^{\infty} c_n$ converges.

Problem B (proposed by Simone Di Marino)

Determine all pairs (a,b) of positive integers such that there are only finitely many positive integers n for which n^2 divides $a^n + b^n$.

Problem C (proposed by Hendrik Lenstra)

Let $f \in \mathbb{Z}[X]$ be a monic polynomial, and let R be the ring $\mathbb{Z}[X]/(f)$. Let U be the set of all $u \in R$ satisfying $u^2 = 1$. Show that U has a ring structure with the following properties: the *zero element* is 1, the *identity element* is -1, the *sum* of two elements in U is their product in R, and the *product* $*$ in U is such that for all u, v, s, t in U the identity $u * v = s * t$ holds in U if and only if

$$(1-u)(1-v) = (1-s)(1-t)$$

holds in R.

Edition 2012-1 We received solutions from: Charles Delorme (Paris), Pieter de Groen (Brussels), Alex Heinis (Hoofddorp), Ruud Jeurissen (Nijmegen), Thijmen Krebs (Nootdorp), Paolo Perfetti (Rome), Merlijn Staps (Leusden), Roberto Tauraso (Rome), Sep Thijssen (Nijmegen), Rohith Varma (Chennai), Traian Viteam (Montevideo) and Hans Zantema (Eindhoven).

Problem 2012-1/A Let m and n be coprime positive integers. Let Γ be the graph that has the disjoint union $\mathbb{Z}/n\mathbb{Z} \sqcup \mathbb{Z}/m\mathbb{Z}$ as vertex set and that has for every $1 \leq i \leq m+n-1$ an edge connecting $i \pmod n$ and $i \pmod m$. Show that Γ is a tree.

Solution We received solutions from Charles Delorme, Pieter de Groen, Alex Heinis, Ruud Jeurissen, Thijmen Krebs, Merlijn Staps, Roberto Tauraso, Sep Thijssen, Rohith Varma, Traian Viteam and Hans Zantema. The book token goes to Hans Zantema. The following solution is based on that of Sep Thijssen.
Note that Γ has $m+n$ vertices and $m+n-1$ edges, so it suffices to show that Γ is connected. Without loss of generality we may assume $n \geq m$. Then note that for all $1 \leq i \leq m-1$, there is an edge connecting $i \pmod m$ to $i \pmod n$ and one connecting $i \pmod n$ and $i+n \pmod m$, since $i+n \leq m+n-1$. In particular, $i \pmod m$ and $i+n \pmod m$ are in the same connected component. Thus $n, 2n, \ldots, mn \pmod m$ lie in the same connected component. As n and m are coprime, this implies that $\mathbb{Z}/m\mathbb{Z}$ is inside a single connected component. Moreover, any vertex in $\mathbb{Z}/n\mathbb{Z}$ is connected to at least one of $\mathbb{Z}/m\mathbb{Z}$, hence Γ is connected.

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Problem 2012-1/B Is it possible to partition a non-empty open interval in closed intervals of positive length? Let Δ be a triangle (including its interior) and let $P \in \Delta$ be an interior point. Is it possible to partition $\Delta - \{P\}$ in closed line segments of positive length?

Solution *Solution to the former question.* We received solutions from Thijmen Krebs, Merlijn Staps and Sep Thijssen. The following is based on the submission of Sep Thijssen, who receives the book token.

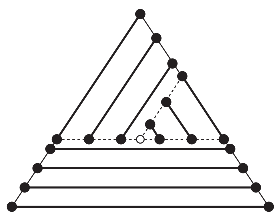
We will prove that it is not possible to partition a non-empty open interval (x_0, x_1) in closed intervals of positive length.

Let a partitioning $\mathcal{P}$ of (x_0, x_1) be given. We will now recursively define a sequence (x_n), starting with the given numbers x_0 and x_1. Consider the interval $[a, b]$ in $\mathcal{P}$ that contains $(x_n + x_{n+1})/2$. If n is even, we put $x_{n+2} = b$, and if n is odd, we put $x_{n+2} = a$.

The sequence $x_0, x_2, \ldots$ is increasing, and the sequence $x_1, x_3, \ldots$ is decreasing. Moreover, we have $|x_{n+1} - x_n| \leq (x_1 - x_0)/2^n$, and thus the sequence $(x_n)_n$ converges. Let $I \in \mathcal{P}$ contain the limit of $(x_n)_n$. Then I contains infinitely many of the x_n, which contradicts the fact that each x_n is an endpoint of the interval of $\mathcal{P}$ it lies in.

Solution to the latter question. We received solutions from Thijmen Krebs, Paulo Perfetti and Roberto Tauraso, Merlijn Staps, Sep Thijssen and Hans Zantema.

There are many ways to partition $\Delta - \{P\}$ in closed line segments of positive length; the following figure depicts one possibility.



Problem 2012-1/C A move on a pair (a, b) of integers consists of replacing it with either $(a+b, b)$ or $(a, a+b)$. Show that starting from any pair of coprime positive integers one can obtain a pair of squares in finitely many moves.

Solution This problem was solved by Charles Delorme, Alex Heinis, Thijmen Krebs and Hans Zantema. The book token goes to Charles Delorme. The following is based on the solution of Thijmen Krebs.

Without loss of generality we may assume that b is odd. Let p be a prime that is congruent to a modulo b and to 3 modulo 4. Such a prime exists by Dirichlet's theorem on primes in arithmetic progressions. By a finite number of moves we move from (a, b) to (p, b).

Similarly, choose a prime $q > b$ that is congruent to 3 modulo 4 and to b modulo p, and move to (p, q).

By quadratic reciprocity either p is a square modulo q or q is a square modulo p, but not both. Without loss of generality we assume that p is a square modulo q. So let x be an integer with $x^2 \equiv p$ modulo q. Let r_1 and r_2 be primes congruent to x modulo q with $r_1 \equiv 1$ and $r_2 \equiv 3$ modulo 4. Then by quadratic reciprocity, q is a square modulo either r_1 or r_2. In any case we find a prime r congruent to x modulo q such that q is a square modulo r.

Since $x^2 \equiv r^2$ modulo q, we can move to (r^2, q). As q is a square modulo r, it is also a square modulo r^2, so we may finally move to (r^2, t^2) for some t.