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# Increasing insightful thinking in analytic geometry

Elsewhere in this issue Ferdinand Verhulst described the discussion of the interaction of analysis and geometry in the 19th century. In modern times such discussions come up again and again. As of 2014, synthetic geometry will not be part of the Dutch ‘vwo – mathematics B’ programme any more. Instead, the focus will be more on analytic geometry. Mark Timmer and Nellie Verhoef explored possibilities to connect the two disciplines in order to have students look at analytical exercises from a more synthetic point of view.

Although analytic geometry is a wonderful technique to prove a variety of theorems in Euclidean geometry in a convincing and easy manner, it rarely provides many insights. Secondary school students often apply it without any consideration of what they are actually doing. We conjecture that this leads to fragmented understanding. Rather than developing an overall picture of the geometric concepts the students are working with, the analytic and synthetic geometry remain isolated domains. This results in limited understanding of the mathematical structures at hand, and a limited set of techniques and strategies for solving exercises from these different domains. Analytic geometry becomes an end in itself; students manipulate formulas without any feeling for the underlying concepts.

Additionally, an analytical approach might sometimes even be much more cumbersome than a synthetic argument. By using analytical techniques for dealing with geometric figures, students sometimes forget about the properties of these objects, resulting in lengthy, unnecessary calculations.

In the context of the first author’s Master’s thesis for his mathematics teaching degree at the University of Twente, we tried to emphasise the underlying concepts of synthetic geometry when covering a chapter on analytic geometry. This was often accompanied by visualisations using the GeoGebra computer programme. The overall goal was to provide students a richer understanding of geometry [12]. More specifically, we were hoping for them to develop richer *cognitive units* [2].

That way, students understand better how different representations of geometric concepts such as ellipses relate, and are able to quickly switch between them. Hence, they might work more efficiently when solving exercises for which a purely analytical approach is unnecessarily difficult.

We already extensively discussed the lesson series and research project that resulted from the ideas above in a previous article [13]. Here, we elaborate more on the theoretical background regarding cognitive units and visualisation of geometric objects. Moreover, we discuss the way in which the results of this research project were put into practice as a workshop during the National Mathematics Days (NWD).

**Underlying school mathematics**

Our research primarily focused on the *ellipse*. This mathematical object can be defined as follows.

**Definition 1.** An ellipse is a set of points that all have the same sum of distances to two given focus points.

**Definition 2.** An ellipse is a set of points that are equidistant from a circle (the *directrix circle*) and a point within that circle.

The first definition is illustrated in Figure 1, the second one in Figure 2.

It is not hard to see that these two definitions coincide. In Figure 1, by definition  $F_1P_1 + P_1F_2 = F_1P_2 + P_2F_2$ ; let this constant

be  $r$ . In Figure 2,  $MP_i + P_iF$  equals the radius of the circle, for both point  $P_1$  and  $P_2$ , and all other points on the ellipse. The ellipse consisting of all points that are equidistant from a point  $F$  and a circle with centre  $M$  and radius  $r$ , therefore coincides with the ellipse consisting of all points with cumulative distance  $r$  to  $M$  and  $F$ . Stated differently,  $M$  and  $F$  are the focus points of the ellipse in Figure 2.

Placing an ellipse in a Cartesian coordinate system with the focus points on the horizontal axis (see Figure 3), we can show that it coincides with the set of points  $(x, y)$  such that  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ . Here,  $a$  is half of the length of the horizontal axis, and  $b$  half of the length of the vertical axis. In Figure 3 this yields  $\frac{x^2}{25} + \frac{y^2}{9} = 1$ . Interestingly, such an analytical representation relates in several ways to the

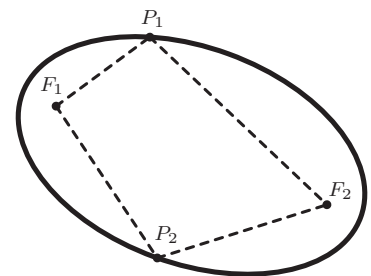


Figure 1 Equal cumulative distance to two focus points

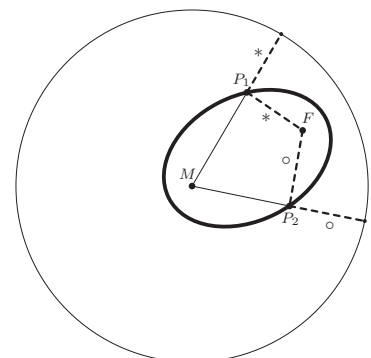


Figure 2 Equal distance to circle and point

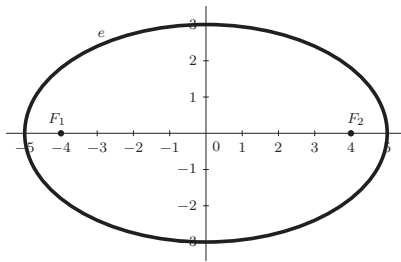


Figure 3 An ellipse in a coordinate system

synthetic definitions discussed above. For instance,  $2a$  corresponds to the radius of the directrix circle, and  $2\sqrt{a^2 - b^2}$  is the distance between the focus points.

We expected proficiency in such conversions between the analytical and the synthetic domain to increase understanding and insight, helping students solve exercises more effectively and efficiently.

### Theoretical framework

In this study we investigated students' cognitive items with respect to geometric objects. In particular, we assessed the effects of a teaching method based on visualisation and synthetic geometry on these units. Hence, this section provides an overview of the theory regarding cognitive units and visualisation.

#### Cognitive units

The human brain is not capable of thinking about many things at once. Complicated activities such as mathematical thinking therefore have to be made manageable by abstracting away unnecessary details and focusing on the most important aspects [2]. The term *cognitive unit* originated from this idea:

"A cognitive unit consists of a cognitive item that can be held in the focus of attention of an individual at one time, together with other ideas that can be immediately linked to it." [10]

The 'cognitive item' mentioned here could be a formula such as  $a^2 + b^2 = c^2$ , a fact such as  $10 + 3 = 13$  or a mental image of an ellipse. The connectivity between cognitive items and related ideas depends on the degree of understanding. For instance, most people would probably immediately relate  $3 + 4$ ,  $4 + 3$  and  $7$ , and hence have strong connections between these cognitive items. They can then be considered as a single cognitive structure: a cognitive unit.

Barnard and Tall emphasise the importance of *rich* cognitive units, having strong internal connections between different objects or representations of objects, and leading to powerful ways of thinking. In our case, several different characterisations of the ellipse are

considered. Initially, such characterisations will probably not be strongly connected in the students' brains. Later on, rich cognitive units might develop, allowing the students to perceive the characterisations as different representations of the same object. This is expected to yield more efficiency and understanding.

*Compression to rich cognitive units.* Rich cognitive units do not develop out of thin air. At first, a student will have a fragmented understanding of a new concept. Then, several different approaches might be needed to obtain a full understanding. However, once a concept has been fully understood, a significant *mental compression* can often be observed. Thurston explains how this results in a complete mental perspective — although at first obtained by a long process — to be easily used as part of a new mental process [11].

The notion of compression is applied on the one hand for the compression of knowledge into small cognitive items [3], and on the other hand for the way in which different cognitive items are coupled into strongly-connected cognitive units [10]. Since both processes yield richer cognitive units, we do not distinguish between these two meanings.

*Causing compression.* In order to induce compression, brain sections have to be connected to such an extent that addressing one of them also activates the others. After all, this makes the combined knowledge and understanding of these sections function together as a single cognitive structure [10].

More specifically, compression can be brought about in several different ways [9]. A student could categorise concepts or perform thought experiments, leading to connections between properties of those concepts. Repeatedly practising certain procedures until they are automated may also yield rich cognitive units. Finally, compression can be induced by *abstraction*: introducing symbols or names. Gray and Tall indeed indicate that we can only effectively talk about phenomena once they have been given a name [3]. As this compresses them to a cognitive unit, it enables us to think about them in a more sophisticated manner.

#### Visualisation

In this study, the underlying concepts from synthetic geometry were often visualised using GeoGebra, a computer programme for dynamic geometry [4]. The geometric objects under consideration indeed perfectly fit dy-

namic visualisation. For instance, we can easily use an equation for an ellipse and a slider determining its parameter  $a$ , to teach students this parameter's effect on the ellipse.

Scientific literature indicates that visualisation may improve mathematical understanding, although this does not necessarily have to happen. Stols explains how the use of IT — more specifically, GeoGebra and Cabri 3D — only positively affects geometric insights of students that did not have much understanding yet, and even then only marginally [8]. He recommends to deploy applications such as GeoGebra to improve visualisation skills and conceptual understanding, and enable students to discover important relations. However, these programmes should not be expected to improve reasoning skills. We indeed only used GeoGebra for visualisation and to observe connections between concepts.

Langill also describes that software like GeoGebra should mainly be used as a supplement to non-technological sources, such as books [6]. She noticed that distance measuring and point dragging are among the most powerful applications of dynamic geometry. Therefore, we indeed combined visualisations with additional exercises, and extensively applied dragging and measurements to illustrate geometric properties.

Other researchers confirmed that technology can help students discover connections between different representations of the same concept, but also noticed that it should not be deployed too early [1]. They found that visualisations should be linked directly to knowledge that the students already possess, to avoid frustration and misconceptions. We therefore only used GeoGebra to clarify concepts the students were already familiar with, avoiding this pitfall.

Despite the potential merits of dynamic geometry software, it is still not used very often. Stols and Kriek report that a negative attitude towards the added value of such software, as well as a lack of confidence in their own technical skills, prohibit teachers from using applications like GeoGebra [14]. Zhao, Pugh, Sheldon and Byers also reached this conclusion, and observed that teachers have to take small evolutionary steps when introducing ICT in the classroom; a revolutionary approach would only lead to failure and frustration [15].

In this study, GeoGebra was only used by the teacher. Obviously, it is also possible to have the students play with the application. Although this is indeed expected to help students discover geometric theorems [7] or understand geometric transformations [5], we

only applied GeoGebra for demonstrations. After all, we did not focus on developing new geometric skills, but more on the application of available geometric knowledge in the context of analytic geometry.

### This study

We performed our study in a vwo 5 mathematics D class at the Stedelijk Lyceum Kotenpark in Enschede. Since this class consisted of only four students (for privacy reasons all addressed by 'he' in this article), we were able to observe the students in much detail and question them individually. The researcher taught Chapter 14 of the *Getal & Ruimte* vwo D<sub>4</sub> method. This chapter covers symmetry, parametric equations and difference quotients, based on parabolas, ellipses and hyperbolas.

We tried to encourage the students to focus on connections between synthetic and analytic geometry in three different ways: (1) by giving additional explanations — often accompanied by GeoGebra visualisations — to make students aware of what they are doing, (2) by discussing how several analytical exercises from the book can be solved more easily using geometric reasoning, and (3) by introducing a number of new exercises for the students to practice these skills on. We refer to [12–13] for an extensive description of the lesson series.

Semi-structured interviews before and after the lesson series have shown quite a different effect on each of the four students. For one of them, the focus on synthetic geometry seemed to work out poorly. This student

showed only limited knowledge and insight, both before and after the lesson series. He preferred to rely on an analytical approach, and already declared upfront to rather just calculate than think of a smarter way to solve an exercise. Additionally, he often indicated to not have much confidence in his own mathematical understanding, explaining his preference for structured rules and procedures.

The other three students were much more enthusiastic, and showed a positive attitude towards the new way of approaching analytic geometry. They most liked the feeling of deeper understanding, as well as the simplicity to achieve results. One student indeed showed considerably more insight during the post-test. He switched rapidly between different representations of the same concept, for instance by using symmetry for an analytical exercise and by combining both definitions of the ellipse in a smart manner. Additionally, he often first took a moment to think before relying on calculations, and showed growth in his associations with geometric concepts.

The other two students showed slightly less progress, but still improved visibly. They were able to identify more representations and more often applied geometric concepts such as symmetry. Interestingly, it appeared that some insights were present, but only surfaced after considerable encouragement. This indicates that certain connections between cognitive items have been made, but also that more practice is needed to enable fast switching between the accumulated knowledge from different domains.

### National Mathematics Days

To share our findings with a larger group of teachers, we conducted a workshop during the most recent National Mathematics Days ([www.fisme.science.uu.nl/nwd](http://www.fisme.science.uu.nl/nwd)). There appeared to be quite some interest in our topic; teachers were happy to discuss a more insightful manner of working with analytic geometry.

After a short introduction of the subject, the teachers were asked to work on some of the exercises the students also tried to solve during their post-test. They intensely calculated and discussed, and appeared to pursue many different approaches. We found that they did not always fully use all available data and possible connections to other representations. The determination to solve the difficult exercises, however, was inspiring. Such an attitude would benefit every student!

The teachers asked many questions about the translation from our ideas to the classroom: how can we make students follow our approach, combining different representations and thinking before computing? As we mentioned before, frequent practice seems to be key. The workshop participants were pleased to hear and experience a creative way of addressing synthetic geometry in the current mathematics curriculum.

More details on the lessons and exercises can be found in [13]. For an extensive description of the research project, we refer to [12]. Both articles, as well as all material used at the NWD, can be found at <http://fmt.cs.utwente.nl/~timmer/research.php>. ←

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