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Perspectives on the legacy of Poincaré in the field of dynamical systems

Henri Poincaré (1854–1912) has had a tremendous influence on the development of mathematics and mathematical physics, the traces of which can be seen till the present day. This paper by Henk Broer attempts to highlight aspects of this legacy and its influence on certain more recent developments, focusing on the field of dynamical systems. His various approaches have invariably led to fruitful subfields, where geometry, algebra and later also measure theory are effectively at work in combination with analysis and calculus.

Poincaré lived in the happy days when the boundaries between mathematics and physics were still very permeable. His great contributions to the development of mathematical physics, e.g. to special relativity theory, are discussed elsewhere, e.g., see [88], and the present paper focuses on dynamical systems. Evidently there are many interconnections between these various fields. Poincaré has been extremely generous with his ideas and his legacy is large and fruitful, a legacy from which the current discipline of dynamical systems greatly profits. The beginning of these developments consisted of the theory of ordinary differential equations, mainly applied to problems related to celestial mechanics.

One main, general characteristic of Poincaré's approach to any of his fields of interest, is his open-mindedness to all kinds of mathematics and his readiness to connect these several fields. Regarding dynamical systems, he extended the then dominant calculus- and analysis driven-approach by including geometry and algebra and also certain considerations close to probability. To some extent this holds for topology as well, although this branch of mathematics was still in its infancy. In fact Poincaré himself has stood at its cradle when developing his 'analysis situs', treated elsewhere in this volume. For reasons of readability in this paper we deliberately maintain an anachronistic style; for a more faithful translation of Poincaré's diction into more modern terminology, we refer to, e.g., [88].

In particular we shall discuss elements of Poincaré's [63, 65] and of his monumental [69], also see [73]. Already in [65] the idea of a phase portrait was introduced, where the attention is not focused on the computation or approximation of one single solution, but where his geometric organization of all solution curves in a state space is being considered. Also we mention the so-called Poincaré map with respect to a transverse section is introduced in [69], restricting to a fixed energy-level, useful for similarly organizing the dynamics of a two-degrees-of-freedom system. (Poincaré sections and maps are generally defined also for dissipative systems: the sections then should have co-dimension 1 in the state space.) Here the concept of homo- or heteroclinic orbit occurs, which can give rise to a phenomenon today called tangle. In the analytic approach, tangle implies the divergence of certain perturbation series, and in later days was related to chaos. Another geometric tool is the index of a singularity of vector fields (now called Poincaré–Hopf index) as well as its relationship with the Euler characteristic of the domain or surface at hand. We also briefly touch upon the theorems of Poincaré–Bendixson and Poincaré–Birkhoff. Although many applications of these ideas occur in the conservative dynamics of celestial mechanics, also the general qualitative phenomenon now called Hopf bifurcation (also called phenomenon of Poincaré–Andronov [4]) forms one of the geometric discoveries. The Poincaré recurrence theorem [68] and the relationship with statistical mechanics will also be discussed.

Celestial mechanics

According to Leonardo da Vinci mechanics is the paradise of mathematicians. Between 1700 and 1900 this assertion surely holds for celestial mechanics, which during four centuries has exerted an enormous impact on the development of science and technology.

Remarks.

– The pièce de résistance of mathematical power in celestial mechanics surely was

Photo: Dornac

Poincaré in his study

the discovery of Neptune in 1846, predicted from aberrations in the orbit of Uranus. In these Newtonian computations Adams and Leverrier played a role [46], while also Bessel made some contributions.

- To illustrate the accuracy of astronomical observations and computations we mention the perihelion precession of Mercury, established at a value of $5600'' = 1.5556^0$ per century. Computations have shown that, within the classical Newtonian mechanics, the perturbations of the other planets account for $5557'' = 1.5436^0$ per century. The difference of $42.98''$ can be explained by Einstein's general theory of relativity [53]. This amount of accuracy was already attained around 1900.

Stability of the solar system

Since Newton, Euler and Laplace a central question has been whether the solar system as witnessed and reported by mankind is 'stable' in the sense that the observed multiperiodic motion will persist for ever. Here one may think of the effectiveness of the Gregorian leap year convention (give or take a corrective leap second now and then) or the prediction of the Easter dates and ask how all this works out in the perpetually long run. The solar system itself is quite large and subsystems of it often have been described in terms of three bodies: say the Earth-Moon-Sun or a Sun-Jupiter-asteroid system. Moreover, mathematical simplifications exist like the Restricted Planar Three Body Problem (RPTB), where two primary bodies move uniformly in circles and where the mass of the third body is considered to be small enough not to effect the motion of the primaries and where everything takes place in one fixed plane.

The main mathematical techniques for a long time came from perturbation theory: that is the analysis of a given motion as a perturbation of a more well-known one, say, in terms of one or several Kepler ellipses. One major tool here consists of series expansions.

Analytic methods, normal forms

Poincaré's thesis [63] contributed a lot to this classical perturbation theory. One part of his legacy consists of a local normal form theory, where the series expansions of a vector field (system of ordinary differential equations) in a equilibrium is duly simplified. For later developments also compare Birkhoff [13], developments that extend till today. This simplification facilitates the perturbation analysis based on the series, where the simplified (or normalized) lower-order part forms the unperturbed system and the higher-order terms the perturbation. One important aspect of the simplification often is given by a toroidal symmetry that turns the lower order part of the series into an integrable system. The integrable approximation in turn can be reduced to lower dimensions as is common in classical mechanics, for a more modern point of view compare Arnold [3]. As we shall see later, such series generically do not converge. (Generically they diverge, where the space of real analytic systems is endowed with the compact-open topology on holomorphic extensions [19, 35, 74, 83].) Needless to say however, they always can serve asymptotic purposes. (At the time this was a controversial issue: convergence was considered essential by mathematicians like Cauchy and Weierstraß.)

In [69] a similar problem is addressed, but now the aim is to continue a periodic solution in terms of a perturbation parameter, for Poincaré's comments on these Lindstedt series see [66–67]. Nowadays one generally speaks of Poincaré–Lindstedt series. Convergence of such a series in general can be obtained as an application of the implicit function theorem or similar methods.

Remarks.

- I like to mention the celestial mechanics computations by De Sitter on the $1:2:4$ orbital resonance of the Galilean Jupiter satellites Io, Europa and Ganymede, that leads to a libration in their motion. This phenomenon was observed by David Gill (Cape Town) and De Sitters computations started around 1900 leading to a PhD thesis under Kapteyn. Later he again spent a lot of time on this subject [81–82], using Poincaré's normalized resonant series expansion [63, 69]. (It seems that De Sitter himself esteemed this celestial mechanics project higher than his work on a global solution for the equations of general relativity, later on coined as the De Sitter universe.)
- For more historical details and references see the very readable De Sitter biography by Guichelaar [29].

Example 1 *(Holomorphic linearization).* To fix our thoughts mathematically, we consider the problem of holomorphic linearization that runs as follows, see [64]. Also see [4, 52]. Given is a holomorphic germ $F : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ of the form

$$F(z) = \lambda z + f(z), \tag{1}$$

where $f(0) = f'(0) = 0$ and the question is whether a bi-holomorphic diffeomorphism $\Phi : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ exists, such that

$$\Phi \circ F = \lambda \cdot \Phi, \tag{2}$$

i.e., that linearizes F. Here λ is a complex parameter. A brief computation shows that a formal solution $\Phi(z) = z + \sum_{j\geq 2} \phi_j z^j$ exists in terms of the Taylor series of f, if and only if $\lambda \neq 0$ and $\lambda \neq e^{2\pi i p/q}$ for all $p \in \mathbb{Z}$ and $q \in \mathbb{N}$, i.e., if λ is not a root of unity. Indeed, it easily can be shown that the coefficient ϕ_n

contains a factor

$$\frac{1}{\lambda(1-\lambda)\cdots(1-\lambda^n)}. \qquad (3)$$

The question then is whether or not the series converges. In the hyperbolic case $0 < |\lambda| \neq 1$ this question is answered affirmatively by Poincaré [64]. For an elaborate discussion, including Poincaré's elegant, geometric solution of this hyperbolic case, see [4]. In the elliptic case where λ belongs to the complex unit circle, the exclusion of λ from being a root of unity is not enough for convergence. Indeed, even in this case the denominators in (3) accumulate on zero, which forms an example of the notorious small divisor problem. Sufficient for convergence are Diophantine conditions on λ, which require that for positive constants τ and γ inequalities

$$\left|\lambda - e^{2\pi i \frac{p}{q}}\right| \geq \frac{\gamma}{q^\tau}, \qquad (4)$$

hold true for all $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. The corresponding subset of λ's in the complex unit circle is both meagre and of full measure. This solution was given by Siegel in 1942, compare with [80]. In his 1969 thesis Bruno extended this sufficient condition in terms of the decay rate of continued fractions; the latter Bruno condition was proven also necessary by Yoccoz [91], for which he received a 1994 Fields medal. This is a highly successful part of mathematical research, that surely belongs to Poincaré's mathematical legacy.

Small divisors

In the conservative dynamics of classical mechanics, including celestial mechanics, small divisor problems occur a lot. As in the above example this phenomenon is related to a dense set of resonances where a certain series is not even defined. The situation can be described as follows, for instance by Moser [57]. In the n-dimensional space of frequencies $\omega_1, \omega_2, \dots, \omega_n$ a resonance occurs if for some integer vector $\mathbf{k} = (k_1, k_2, \dots, k_n) \in \mathbb{Z}^n \setminus \{0\}$ the relation

$$k_1\omega_1 + k_2\omega_2 + \cdots + k_n\omega_n = 0 \qquad (5)$$

holds. For fixed $\mathbf{k}$ equation (5) defines a hyperplane in the frequency space and the union of these planes over all $\mathbf{k} \in \mathbb{Z}^n \setminus \{0\}$ forms the dense resonance web. Certain perturbation series of the form

$$\sum_{\mathbf{k}\in\mathbb{Z}^n\setminus\{\mathbf{0}\}} \frac{c_{\mathbf{k}}}{k_1\omega_1 + k_2\omega_2 + \cdots + k_n\omega_n} \cdot e^{2\pi i(k_1x_1+k_2x_2+\cdots+k_nx_n)},$$

show up, where the constants $c_{\mathbf{k}}, \mathbf{k} \in \mathbb{Z}^n$ a priori only satisfy a certain decay condition as $|\mathbf{k}| \to \infty$. As in Example 1 exclusion of the resonances gives formal existence of the series, but convergence still is problematic due to small divisors.

Poincaré recognised the problem that was solved later by Kolmogorov [41–42] and many others, see below. This problem concerns the persistence of quasi-periodic motion as these occur in simplified approximations in which, e.g., uncoupled Keplerian motions show up. This perturbation problem was at the heart of the contest called by king Oscar II of Sweden.

Remarks.

- This history is world-famous, for a very readable and historically detailed account see June Barrow-Green [10]. Summarizing we recall that Poincaré's essay, although containing a wealth of ideas, contained one essential mistake, namely the claim that the perturbation series do converge.
- Poincaré discovered (triggered by a query of Phragmén) and repaired the mistake, concluding that the series have to diverge for a significant set of frequencies. Mittag-Leffler let him use the prize money to buy the volumes of the *Acta Mathematica* that were already published. His divergence proof, published in the article [68] in the same journal, uses geometric methods that we will address now.

Tangle and divergence

The restricted planar three body (RPTB) problem ends up with a two-degrees-of-freedom Hamiltonian system, which lives in a four-dimensional state space. Then, restricting to a three-dimensional energy hypersurface we can consider a section (or slice) transverse to the dynamics. Following the integral curves of the system in turn gives rise to a two-dimensional Poincaré map, compare the sketch in Figure 1 for some orbits. In the present setting of classical mechanics such a map preserves area. If the perturbation series at hand were convergent the orbits of this map would nicely trace the integral curves of a vector field. The presence of tangle however prevents this convergence, as Poincaré himself concluded, compare with, e.g., Takens [84].

What then is tangle? To understand this we recall that a hyperbolic fixed point p of a map has both a stable manifold $W^s(p)$ and an unstable manifold $W^u(p)$, which in the present planar case are smoothly immersed curves. The points of $W^s(p)$ converge to p under forward iteration and similarly the points of $W^u(p)$ by backward iteration. For two of such saddle points p_1 and p_2 it may happen that the intersection $W^u(p_1) \cap W^s(p_2)$ is nonempty. The intersection consists of point heteroclinic to p_1 and p_2. In the case where $p_1 = p_2$ the intersection points are called homoclinic (Poincaré himself speaks of double-asymptotique [88]). For a vector field, by uniqueness of solutions, this would imply coincidence of $W^u(p_1)$ with $W^s(p_2)$. For a map however, this intersection generically is transverse, again see [19, 35, 74, 83]. In Figure 1 a sketchy impression is given of tangle, for a realistic impression how the tangle explodes near a saddle point see Figure 3 (right).

Figure 1 Hetero- and homoclinic 'tangle' in the stable and unstable manifolds of a periodic orbit, sketched for a planar iso-energetic Poincaré map as this occurs in a suitable three-body problem.

From that time on the mainly analytical investigation of dynamical systems has been extended by considering the geometrical organisation of the entire state space (or phase space). This involved the study of equilibria, periodic solutions, invariant manifolds like stable and unstable manifolds and invariant tori. At the same time the mathematical assertions became more qualitative.

Geometric ideas

We introduce a few geometric ideas and results introduced and obtained by Poincaré. This includes the rotation number and the Poincaré–Hopf index theory for equilibrium points of vector fields in the two-dimensional case; the higher dimensional case was added later by Heinz Hopf. Also we deal with the Poincaré–Birkhoff and the Poincaré–Bendixson theory.

Rotation number of circle homeomorphisms

The rotation number for an orientation preserving homeomorphism of the circle $\Phi : \mathbb{S}^1 \to \mathbb{S}^1$, first considered in Poincaré [65], is defined as the average rotation

$$\varrho(\Phi) = \frac{1}{2\pi n} \lim_{n\to\infty} \left(\tilde{\Phi}^n(x) - x\right) \mod \mathbb{Z},$$

where $\tilde{\Phi} : \mathbb{R} \to \mathbb{R}$ is a lift of Φ. This notion depends neither on the point $x \in \mathbb{S}^1$ nor on the choice of the lift $\tilde{\Phi}$ and moreover is invariant under topological conjugation.

Rationality or irrationality of the rotation number. If Φ has a periodic orbit of (prime) period q, then $\varrho(\Phi) = p/q$ for some p coprime with q. Conversely $\varrho(\Phi)$ is rational implies that Φ has a periodic orbit. On the other hand, if $\varrho(\Phi)$ is irrational and Φ is of class C^2, then Φ is topologically conjugated to a rigid rotation. These results go back to Denjoy [21]. For details also see Devaney [22] and Nitecki [59].

In the case where the rotation number is Diophantine there exist smooth conjugations with rigid rotations, compare with Example 1. The perturbative version of this by now is a standard result in Kolmogorov–Arnold–Moser theory as described further on, but the non-perturbative case certainly is not, see Herman [33–34].

The Poincaré–Birkhoff theorem on periodic orbits. The Poincaré–Birkhoff fixed point theorem fits in the theory of area preserving planar maps as met before, compare with the settings of Figures 1 and 3. Consider such a planar map Φ with two invariant circles with rotation numbers $\varrho_1 < \varrho_2$, bounding an annulus. The Poincaré–Birkhoff theorem then asserts that for any rational number p/q with $\varrho_1 < p/q < \varrho_2$ there exists a Φ-periodic orbit of rotation number p/q. Also see the contribution of Verhulst to this volume. For proofs see Poincaré [72] and Birkhoff [12]. A simple proof can be given in the case where Φ is a perturbed twist-map, see Birkhoff [13]. See Arnold et al. [5] and Moser [56] for presentations of this proof based on the implicit function theorem. For further developments we refer to [31–32, 40].

Poincaré–Hopf index theory

This topic connects the possible dynamics of any system to the global topology of the manifold on which it is defined, see [65]. Consider a vector field X on a closed surface M. If $p \in M$ is an isolated equilibrium point (also called rest point or singularity) of X we can define the index $\text{ind}_p X$ by taking any small simple curve $t \in \mathbb{S}^1 \mapsto \gamma(t) \in M$ around p and considering the well-defined direction field $1/(|X(\gamma(t))|) \cdot X(\gamma(t))$. Here $\mathbb{S}^1$ denotes the unit circle. This induces a map $\mathbb{S}^1 \to \mathbb{S}^1$ and we define $\text{ind}_p X$ as its winding number.

Remarks.

- Locally one can define the closed 1-form

  $$đ\varphi = \frac{X_2\, \mathrm{d} X_1 - X_1\, \mathrm{d} X_2}{X_1^2 + X_2^2},$$

  where

  $$X(x_1, x_2) = X_1(x_1,x_2)\partial_{x_1} + X_2(x_1,x_2)\partial_{x_2};$$

  in this case we can express $\text{ind}_p X = \frac{1}{2\pi}\oint_\gamma đ\varphi$, compare with [2, 51].
- This definition was generalized to higher dimension by Heinz Hopf, using Brouwer's notion of degree, see [38].

There is an interesting connection with the global topology of the surface, given by the formula

$$\sum_p \text{ind}_p X = \chi(M),$$

where $\chi(M)$ is the Euler-characteristic of M, also compare with Lakatos [44]. One consequence of this is the well-known 'hairy ball' theorem that any vector field X on the 2-sphere $M = \mathbb{S}^2$ must have at least one singularity since $\chi(\mathbb{S}^2) = 2$.

The Poincaré–Bendixson theorem

Another interesting result [65] concerns the surfaces $M = \mathbb{R}^2$ or $M = \mathbb{S}^2$, dealing with the possibilities of the asymptotic dynamics as time goes to infinity. The corresponding set is the ω-limit set of a point $p \in M$ given by

$$\omega(p) = \{y \in M \mid \lim_{j\to\infty} \Phi^{t_j}(p) = y \text{ for a sequence } \{t_j\}_{j=1}^\infty \text{ with } \lim_{j\to\infty} t_j = \infty\}.$$

The Poincaré–Bendixson theorem asserts that on $M = \mathbb{R}^2$ or $M = \mathbb{S}^2$ the ω-limit sets of vector fields with a finite number of equilibria can only be

- equilibria,
- limit cycles,
- or graphs of equilibrium points and their stable and unstable manifolds,

see Poincaré [65] and Bendixson [11]. This result often is used to prove the existence of a (i.e., at least one) limit cycle in the absence of any equilibria. Compare with Palis and De Melo [61] or with Verhulst [87].

Multi- or quasi-periodicity

Multi-periodic and chaotic dynamics have obtained a lot of interest in the second half of the 20th century, an interest that surely belongs to the legacy of Poincaré. A central role was played by Kolmogorov [41–42]. To explain this we again address the stability of the solar system, the prize essay question.

Integrable systems

Multi- or quasi-periodic dynamics were very well-known from the completely integrable systems of classical mechanics. In a Hamiltonian system with n degrees of freedom an open and dense part of the entire state space then is foliated by n-dimensional Lagrangian invariant tori. Examples are given by systems of uncoupled Keplerian motion. Locally an integrable system is described by action-angle coordinates $(I, \varphi) = (I_1, I_2, \ldots, I_n, \varphi_1, \varphi_2, \ldots, \varphi_n)$, in which the symplectic form reads $\mathrm{d}I \wedge \mathrm{d}\varphi = \sum_{j=1}^n \mathrm{d}I_j \wedge \mathrm{d}\varphi_j$. Compare with Arnold [3]. The Lagrangian tori are parametrized by I varying over an open piece of $\mathbb{R}^n$. (The entire union of Lagrangian tori forms a bundle, for a nice description see Duistermaat [23].)

The motion on such a torus $\mathbb{T}^n = (\mathbb{R}/2\pi\mathbb{Z})^n$ then is generated by a constant vector field

$$\begin{aligned} \dot\varphi_1 &= \omega_1 \\ \dot\varphi_2 &= \omega_2 \\ &\vdots \\ \dot\varphi_n &= \omega_n, \end{aligned}$$

or $\dot\varphi = \omega$ for short, where $\omega = \omega(I)$. Usually such a motion is called conditionally periodic or multi-periodic [3, 18]. Here the resonance web (5) again plays a role. Indeed, in the complement of the web any individual integral curve densely fills the invariant n-torus that it is part of. Usually this form of multi-periodic is called quasi-periodic. Within the web the n-tori are foliated by lower-dimensional invariant tori with similar dense orbits and at n-fold resonance points the torus is foliated by periodic orbits.

The matter of stability of the solar system has boiled down to the question to what extent this geometric structure persists under small perturbations that destroy the toroidal symmetry. A simple-minded, yet significant, setting of this problem runs as follows. A

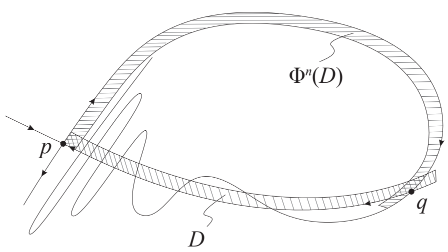


Figure 2 Two-dimensional diffeomorphism Φ with a saddle point p and a transversal homoclinic intersection q. The set D, homeomorphic to the unit square, by a well-chosen iterate Φ^n covers D in a horseshoe-like way.

completely integrable approximate system is formed by considering the Newtonian planetary motion around the sun, neglecting the interactions between the planets. This approximation is formed by the uncoupled dynamics on a number of Keplerian ellipses. One first obstacle to prove stability for the full system, i.e. including the interaction between the planets, is formed by rather strong orbital resonances that occur in the integrable approximation, which has led to a similar but more sophisticated set-up.

Kolmogorov–Arnold–Moser theory

The persistence problem of tori in nearly integrable Hamiltonian systems again has to deal with small divisors, it is basically the same problem that Poincaré faced in his [68–69], also compare with [80]. We present the solution given by Kolmogorov [41–42].

Given suitable constants $\tau > n - 1$ and $\tau > 0$, Diophantine non-resonance conditions are introduced of the form

$$|\langle \mathbf{k}, \omega \rangle| \geq \frac{\gamma}{|\mathbf{k}|^{\tau}}, \qquad (6)$$

for all $\mathbf{k} \in \mathbb{Z} \setminus \{\mathbf{o}\}$, compare with (4). This defines a closed, nowhere dense set of positive Lebesgue measure, tending to full measure as $\gamma \downarrow 0$. Moreover a non-degeneracy condition — later named after Kolmogorov — is introduced, requiring that the frequence map $I \mapsto \omega(I)$ is a local diffeomorphism.

Kolmogorov's theorem now roughly reads as follows. Given Kolmogorov non-degeneracy, for sufficiently small perturbations of the integrable system, the Diophantine tori are persistent, their union forming a subset of the state space having positive Liouville measure.

Remarks.

- Refinements and further proofs have been added by Arnold [1] and Moser [54], also see [5, 7] and [55–58], for which reason we now speak of Kolmogorov–Arnold–Moser (or KAM) theory. A similar result holds when restricting to a fixed level of energy, this is the iso-energetic KAM theorem.
- In fact the quasi-periodic motion on the unperturbed and perturbed tori is smoothly conjugated, where in the I-direction smoothness has to be interpreted in the sense of Whitney [89].
- The KAM theory extends to other classes of systems, such as the general dissipative class, reversible or volume-preserving systems, etc., see [55], for an overview and many references see [16–17].

A second obstacle for proving the stability of the solar system in this way, is formed by the fact that the perturbations are far too large for applications of KAM theory. Recently rumours go that the inner solar system might very well be chaotic, see Laskar et al. [43]. Based on an estimate of Lyapunov exponents the first major disasters in this respect may be expected on a time-scale of the order of 100 million years... This is relatively soon as compared to the age of the solar system.

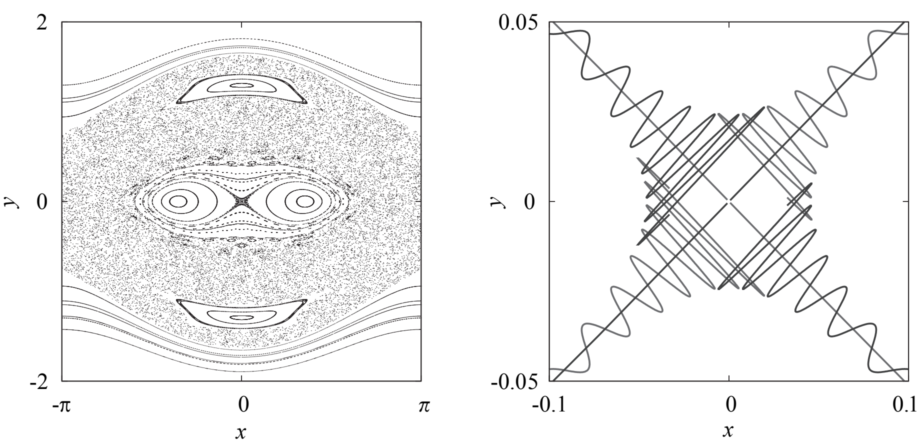


Figure 3 Chaos in a model of the swing in a 1 : 2 resonance, e.g., see [18]. Left: Iterates of a Poincaré map (or stroboscopic map) are shown. The invariant circles are within reach by KAM theory. The large cloud of points is part of one single 'chaotic' orbit, which is related to tangle and Smale horseshoes. Right: a few 'switches' of the tangling separatrices near the central, unstable fixed point.

Tangle and the Smale horseshoe

Homoclinic tangle leads to chaos. This is an anachronism, but the description that Poincaré himself gives of Figure 1 explains that he must have had an insight in the enormous complexity of a generic area preserving map ([69], Vol. III):

"Que l'on cherche à se représenter la figure formée par ces deux courbes [...] On sera frappé de la complexité de cette figure, que je ne cherche même pas à tracer. Rien n'est plus propre à nous donner une idée de la complication du problême des trois corps et, en général, de tous les problèmes de dynamique où il n'y a pas d'intégrale uniforme et où les séries de Bohlin sont divergentes."

In fact it is this complexity that prevents the series from converging, compare with Takens [84]. Throughout the 20th century this phenomenon recurred in various examples, where several forms of symbolic dynamics were introduced to describe this, starting with Birkhoff [13] and Cartwright–Littlewood [20]. In the 1960s the topologist Smale [83] introduced a universal model in the form of the horseshoe map, see Figure 2. Often it takes a sufficiently high iterate of the map to turn heteroclinic points, e.g. as in Figure 1, into homoclinic ones. This largely reduced the problem to geometry and symbolic dynamics. The presence of horseshoes already implies a weak form of chaos, since the topological entropy is non-zero [5, 49]. At the end of this paper we will say more about chaos.

Invariant measures

The phase flow of an n degree-of-freedom Hamiltonian system conserves the Liouville volume, which is the nth wedge-power of the symplectic form, compare with [3]. Similarly the restriction of this flow to an energy hypersurface conserves the conditional $(2n - 1)$-dimensional volume. The corresponding iso-energetic Poincaré maps from Figures 1 and 3 inherit a natural area form.

One consequence of measure-preservation is that no point attractors can occur. Below we first discuss the Poincaré recurrence theorem and next indicate a few links with statistical mechanics, where also the subject of ergodicity will come to pass. An important part of the current status quo in the area of dynamical systems rests on these theories, compare with Ruelle [75–76]. This part of Poincaré's legacy runs by Kolmogorov and Sinai [78].

The Poincaré recurrence theorem

The Poincaré recurrence theorem stems from [68], for the formulation below see also [3]. Let D be a set of bounded volume and consider a map $\Phi : D \to D$ that preserves volume. Then for any point $x \in D$ and any neighbourhood U of x there exists a point $y \in U$ such that the iterate $\Phi^n(y) \in U$ for some $n > 0$. A proof can be given as follows. In the infinite sequence

$$U, \Phi(U), \Phi^2(U), \ldots,$$

of sets all have the same volume. Since D has bounded volume, there must be some $k \geq 0$ and $\ell \geq 0$ with $k \neq \ell$, say with $k > \ell$, such that

$$\Phi^k(U) \cap \Phi^\ell(U) \neq \emptyset,$$

which implies that $\Phi^{k-\ell}(U) \cap U \neq \emptyset$. From this the assertion follows by taking $n = k - \ell$ and $y \in \Phi^{k-\ell}(U) \cap U$.

Repeating the argument with U replaced by $\Phi^{n_1}(U)$ with $n_1 = k - \ell$ gives an infinite sequence of instants $n_1, n_2, n_3, \ldots$, for which the iterate comes close to the initial point x.

Remarks.

- Periodic and quasi-periodic motions are recurrent as well as are certain forms of more complicated motions, like ergodic or chaotic motions. See below.
- Recurrence times can be extremely long, a great many orders of magnitude longer than the age of the universe, also see [86]. Below we shall return to this.

Links with statistical mechanics

Boltzmann was a pioneer in statistical mechanics and thermodynamics [14], where he introduced probability in many-particle systems that model gases and fluids. In the ensuing thermodynamics macroscopic quantities emerge like energy, temperature, pressure, entropy, where the latter is a measure for the disorder of the dynamics. The first law of thermodynamics is conservation of energy, while the second concerns the increase of entropy. The relation of the second law with dynamics was puzzling. From the beginning on a leading question has been to what extent thermodynamics is compatible with classical mechanics. Two themes play a special role in this development, namely irreversibility (leading to the well-known 'arrow of time' [24]) and ergodicity.

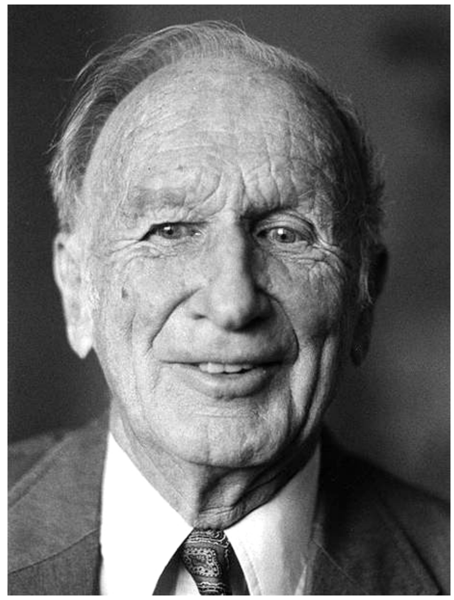
Reversibility. In general classical mechanics is invariant under time-reversal, while thermodynamics is not as a consequence of the second law. This controversy was already noted by Loschmidt [48], the so-called 'Umkehreinwand', for continuation of the discussion see Boltzmann [15]. The answer to the problem lies in statistics: reversal is possible, but highly improbable. (This also 'explains' the extremely long recurrence times noted before. For later considerations see [45].) Zermelo [92] moreover observed that Poincaré recurrence gives an explicit obstruction to irreversibility; this is the so-called 'Wiederkehreinwand'. We now quote Poincaré himself as follows [70], indicating that he was not convinced by these applications of his recurrence theorem:

"Qu'une goutte de vin tombe dans un verre d'eau; quelle que soit la loi du movement interne du liquide, nous verrons bientôt se colorer d'une teinte rosée uniforme et à partir de ce moment on aura beau agiter le vase, le vin et l'eau ne paraîtront plus pouvoir se séparer. (...) Tout cela, Maxwell et Boltzmann l'ont expliqué, mais celui qui l'a vu plus nettement, dans un livre trop peu lu parce qu'il est difficile à lire, c'est Gibbs dans ses principes de la Mécanique Statistique."

Ergodicity. Ergodicity of a particle system also goes back to Boltzmann [14]. Heuristically it expresses that all possible states of the system will be approximated from virtually any given initial state. A mathematical characterisation is that a measure preserving system with evolution $\Phi^t, t \in \mathbb{R}$, is ergodic whenever no disjoint, Φ-invariant subsets A and B of positive measure exist. This indeed expresses that the dynamics of the (particle) system is quite mixing. For stronger mixing conditions also see the ensuing work on ergodic theory of dynamical systems [5, 13, 49, 78].

In statistical mechanics often ergodicity is assumed per hypothesis, restricting to an energy level set. The existence of further integrals, e.g., due to additional assumptions like symmetry, would form an obstruction to the ergodic hypothesis. It was already noted early that stability of the solar system, in the sense that integrals exist, would provide a counterexample. Later it was proven that generically in the dynamics of classical mechanics, apart from the energy, no further integrals occur [19, 74]. However by definition near-integrability is a persistent property.

Kolmogorov [42] (we recall his 1954 closing address of the IMC in the Amsterdam Concertgebouw) noticed a far more serious obstruction to the ergodic hypothesis when initiating KAM theory, see above. Indeed, the persistent occurrence of a union of invariant quasi-periodic of positive measure per energy level [3] leads to a generic violation of ergodic hypothesis in classical mechanics. It seems, however, that this particular obstruction is diminishing when the number of degrees of freedom increases, e.g., see [39].



Edward Norton Lorenz (1917–2008)

Remarks.

- As reported in [30] (a book review is contained in this same volume) there was an interaction between Postma and Poincaré [71] on fluctuations in the entropy where the notions 'entropie grossière' (or 'coarse grained entropy') and 'entropie fine' play a role. The former of these is related to the modern concept of Markov partition [18, 40].
- It should be mentioned here that Einstein's paper [25] on Brownian motion

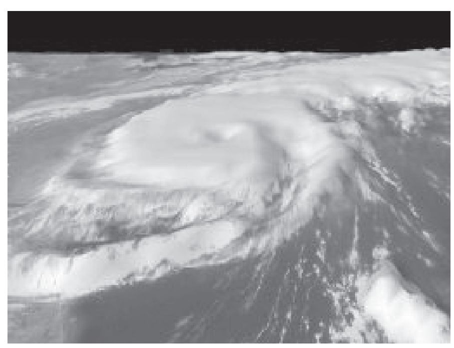


Figure 4 An enormous input to 'chaos theory' came from fields like meteorology; here we see a hurricane.

has a predecessor in Bachelier's PhD thesis [9], written under supervision of Poincaré on an analogous piece of mathematics, applied to financial problems.
- Regarding special relativity there is some controversy on priorities between Lorentz and Poincaré on the one hand, and Einstein on the other. None other than Whittaker [90] attributes special relativity to Lorentz and Poincaré. Also see [88].

Chaos

From the 1970s on the idea of chaos, as a possible property of non-linear dynamical systems that implies a fundamental unpredictability of evolutions, became increasingly important. Needless to say that the development of this part of the theory also belongs to the Poincaré legacy. In a further characterization of chaos an invariant measure comes into play, for instance the measures related to the Liouville volume as discussed before and as these show up in conservative dynamics. In fact one characterisation then is whether the system is ergodic with respect to this. In Figure 3 (left) the large cloud is part of one orbit and one long-standing conjecture is that this orbit densely fills a set of positive area on which the dynamics is ergodic [5].

Remarks.
- Also in the world of dissipative systems chaos has become important. Triggered by Lorenz's ground-breaking paper [47], also see Figure 4, and other examples, the idea of strange attractor [77] came up on which later on invariant measures were constructed which led to ergodicity as a characterization of chaos. One key notion in this respect is the Sinai–Bowen–Ruelle measure, for details see [17–18, 31–32, 40, 78–79].
- Related to this development dynamical invariants were developed, like entropy (both topological and metric i.e. measure theoretical), Lyapunov exponents, etc., that also are important for application in concrete models, also see [26, 60, 74, 87–88].

Concluding remarks

Inevitably in a rather short paper like the present one it is not possible to give a complete description of Poincaré's legacy in the theory of dynamical systems. We have focused on the lines via Birkhoff, Kolmogorov, Sinai and Arnold, for instance tacitly passing by the Russian schools of Lyapunov, Bogolyubov and Andronov. For supplementary information the reader is referred to [27]. After Siegel and Moser the emphasis on celestial mechanics was also diverted to other fields. Smale and Thom [83, 85], both Fields medal winners on variations of the Poincaré conjecture, introduced the concepts of transversality and genericity in the field of dynamical systems. (The Poincaré conjecture is treated elsewhere in this volume. The conjecture was recently solved by the Russian mathematician Perelman.) Among many other things they had a profound influence on the systematic study of singularity theory and bifurcations [6, 8, 85], for another link with chaos also see Palis and Takens [62].

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