Problemen

| Problem Section

This Problem Section is open to everyone; everybody is encouraged to send in solutions and propose problems. Group contributions are welcome.
For each problem, the most elegant correct solution will be rewarded with a book token worth 20 euro. At times there will be a Star Problem, to which the proposer does not know any solution. For the first correct solution sent in within one year there is a prize of 100 euro.
When proposing a problem, please either include a complete solution or indicate that it is intended as a Star Problem. Electronic submissions of problems and solutions are preferred (problems@nieuwarchief.nl).
The deadline for solutions to the problems in this edition is March 1st, 2011.

Problem A (proposed by Gabriele Dalla Torre)
Show that there are infinitely many prime numbers p for which there is a positive integer n with

$$2^{n^2+1} \equiv 3^n \pmod{p}.$$

Also, show that there are infinitely many prime numbers p for which there is no such n.

Problem B (communicated by Marten Wortel)
Let $f\colon \mathbf{R} \to \mathbf{R}$ be a continuous function that has a local minimum or maximum at every point of $\mathbf{R}$. Show that f is constant.

Problem C (communicated by Arne Smeets)
Let $f\colon \mathbf{Q} \times \mathbf{Q} \to \mathbf{Q}$ be a function such that for all $a \in \mathbf{Q}$ the functions $x \mapsto f(a,x)$ and $x \mapsto f(x,a)$ are polynomial functions from $\mathbf{Q}$ to $\mathbf{Q}$. Is it true that f is given by a polynomial in two variables? What if we replace $\mathbf{Q}$ by $\mathbf{R}$?

Edition 2010-2 We received submissions from Pieter de Groen (Brussel), Alex Heinis (Hoofddorp), Thijmen Krebs (Nootdorp), Julian Lyczak (Odijk), Tejaswi Navilarekallu (Amsterdam), and Sep Thijssen (Nijmegen).

Problem 2010-2/A Show that for every positive integer n and every integer $m \geq 2$, we have

$$\sum_{\substack{1 \leq i \leq n \\ m \nmid i}} \lfloor \log_m(n/i) \rfloor = \lfloor n/m \rfloor.$$

Solution This problem was solved by Pieter de Groen, Alex Heinis, Thijmen Krebs, Julian Lyczak, Tejaswi Navilarekallu, and Sep Thijssen. The book token goes to Julian Lyczak.
Since every integer k has a unique representation $k = im^e$ with $m \nmid i$, the function

$$\{(i,e) \in \mathbf{Z}^2 : m \nmid i, 1 \leq i \leq n, 1 \leq e \leq \log_m(n/i)\} \to \{k \in \mathbf{Z} : 1 \leq mk \leq n\}$$

given by

$$(i,e) \mapsto im^{e-1}$$

is a bijection. From this the claimed identity follows at once.

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Oplossingen | Solutions

Problem 2010-2/B Let $b \geq 2$ be an integer. We let $\sigma_b(n)$ denote the sum of the digits in base b of the integer n. Show that we have

$$\lim_{n\to\infty} \sigma_b(n!) = \infty.$$

Solution This problem was solved by Thijmen Krebs and Sep Thijssen. The book token goes to Sep Thijsen.

First note that for all integers $m, n \geq 1$ we have $\sigma_b(m) + \sigma_b(n) \geq \sigma_b(m+n)$, because this is true when m and n are digits. Let $k \geq 1$ be an integer and let $\sigma_{b^k}(n)$ denote the sum of the digits in base b^k of the integer n. Then for any integer $n = \sum_j n_j b^{jk}$ with $0 \leq n_j < b^k$ we have

$$\sigma_b(n) = \sum_j \sigma_b(n_j) \geq \sigma_b\big(\sum_j n_j\big) = \sigma_b(\sigma_{b^k}(n)).$$

We show by induction that for every multiple n of $b^k - 1$ we have $\sigma_b(n) \geq k(b-1)$. For $n = b^k - 1$ equality holds. Assume $n > b^k - 1$ is a multiple of $b^k - 1$. Then also $\sigma_{b^k}(n) < n$ is a multiple of $b^k - 1$, so we have

$$\sigma_b(n) \geq \sigma_b(\sigma_{b^k}(n)) \geq k(b-1)$$

by the induction hypothesis.

For n large enough, the integer $n!$ is divisible by $b^k - 1$, so we have $\sigma_b(n!) \geq k(b-1)$. We conclude

$$\lim_{n\to\infty} \sigma_b(n!) = \infty.$$

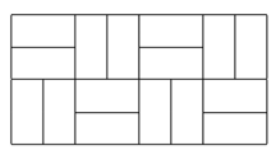
Problem 2010-2/C Two players play a game of n-in-a-row on an infinite checkerboard. The first player plays with white pieces, the second with black pieces. On each move they place one piece on an empty square. The first player to have n consecutive pieces in a row or column wins. For which values of n is there a winning strategy for one of the players?

Solution This problem was solved by Thijmen Krebs. He receives the book token and the solution shown here is based on his submission.

Let us first note that the rules are so that having an extra piece on the board is never a disadvantage. Hence by strategy-stealing, White can at least draw the game for any n.

Let us also note that for $n \in \{1, 2, 3\}$, White has an easy win in n moves.

The next thing we will show is that for $n \geq 5$, Black can force a draw. To do this, we partition the checkerboard in domino-shaped subsets, by repeating the following pattern:



If White places a piece on the board, Black can respond by playing in the same 'domino'. This way, any 5 consecutive pieces in a row or column will always contain both a black and a white piece, so neither player will win.

Let us thus assume $n = 4$. We will show that White has a winning strategy. A position with 3 consecutive white pieces having empty squares on both ends, but without 3 consecutive black pieces elsewhere on the board, can trivially be won by White:

We will now write down a winning stragegy for White by means of a game tree. The moves are written down by the letter 'W' or 'B', indicating whose turn it is, followed by coordinates in $\mathbf{Z}^2$. Without loss of generality, we may assume that White's first move is $\mathrm{W}(3,3)$ and that Black's first move is $\mathrm{B}(x_0, y_0)$ with $x_0 \leq y_0 \leq 3$. We will only list those black moves that prevent White from immediately being able to reach either 4-in-a-row or the type of winning position described above. With this in mind, the following table describes a full winning strategy for White:

Winning strategy for White, starting with $\mathrm{W}(3,3)$

If Black plays $\mathrm{B}(2, y_0)$:								
$\mathrm{W}(3,2)$	$\mathrm{B}(3,1)$	$\mathrm{W}(3,4)$	$\mathrm{B}(3,5)$	$\mathrm{W}(2,4)$	$\mathrm{B}(1,4)$	$\mathrm{W}(4,4)$	$\mathrm{B}(5,4)$	$\mathrm{W}(4,5-y_0)$
					$\mathrm{B}(4,4)$	$\mathrm{W}(1,4)$	$\mathrm{B}(0,4)$	$\mathrm{W}(1,5-y_0)$
	$\mathrm{B}(3,4)$	$\mathrm{W}(3,1)$	$\mathrm{B}(3,0)$	$\mathrm{W}(2,1)$	$\mathrm{B}(1,1)$	$\mathrm{W}(4,1)$	$\mathrm{B}(5,1)$	$\mathrm{W}(4,5-y_0)$
					$\mathrm{B}(4,1)$	$\mathrm{W}(1,1)$	$\mathrm{B}(0,1)$	$\mathrm{W}(1,5-y_0)$
If Black plays $\mathrm{B}(x_0, y_0)$ with $x_0 \leq 1$ and $y_0 \geq 2$:								
$\mathrm{W}(3,2)$	$\mathrm{B}(3,1)$	$\mathrm{W}(3,4)$	$\mathrm{B}(3,5)$	$\mathrm{W}(4,4)$	$\mathrm{B}(x,4)$	$\mathrm{W}(4,5-y_0)$		
	$\mathrm{B}(3,4)$	$\mathrm{W}(3,1)$	$\mathrm{B}(3,0)$	$\mathrm{W}(4,1)$	$\mathrm{B}(x,1)$	$\mathrm{W}(4,5-y_0)$		
If Black plays $\mathrm{B}(x_0, y_0)$ with $y_0 < 2$:								
$\mathrm{W}(3,4)$	$\mathrm{B}(3,y)$	$\mathrm{W}(4,3)$	$\mathrm{B}(x,3)$	$\mathrm{W}(4,4)$