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Interview Alain Connes

# The flashes of insight never came for free

The ‘Fellowship of Geometry and Quantum Theory’ (GQT), one of the four mathematics clusters in the Netherlands, marked the end of its initial four-year funding period with a conference at Nijmegen in June. One of the speakers at this conference was Fields Medallist Alain Connes, who may be regarded as one of the intellectual fathers of the cluster. GQT-members Gunther Cornelissen, Klaas Landsman, and Walter van Suijlekom interviewed Connes on June 29, 2010.

Alain Connes (1947) is among the few mathematicians who created an entire area of math-

ematics. Roughly speaking, Connes’s *noncommutative geometry* [2, 4] is a synthe-

sis and generalization of two seemingly unrelated areas of mathematics, namely operator algebras on Hilbert spaces (see box below) and a branch of differential geometry called *spin geometry* (see box on next page). Both topics emerged from a close interaction of mathematics and quantum physics, which happens to be the central theme of the GQT-

## Noncommutativity

The history of noncommutative geometry goes back to the period 1900–1930, during which both mathematics and physics were revolutionised. In the former, functional analysis emerged (cf. [1, 6]), whilst in the latter quantum mechanics was discovered [12]. The key idea behind functional analysis is to look at functions as points in some infinite-dimensional (topological) vector space, rather than individually, as in classical analysis. A sound physical principle underlying quantum mechanics remains to be found, but the two main mathematical properties of the new theory were as follows. First, in 1925 Heisenberg discovered that whereas in classical physics the *observables* (like position, momentum, and energy) are represented by functions (on a so-called phase space), in quantum mechanics they are (typically infinite-dimensional) *matrices*. In particular, as Heisenberg observed, quantum-mechanical observables no longer commute (under multiplication). Second, in 1926 Schrödinger proposed that *states* of

a physical system (which assign values to observables) are (‘wave’) *functions* (rather than points in phase space).

Heisenberg was a postdoc in Göttingen at the time, where Hilbert ran a seminar on the mathematical structure of the new quantum mechanics. In this context, it was Hilbert’s assistant von Neumann (originally employed to help Hilbert with his work on the foundations of mathematics) who at one stroke saw the connection between functional analysis and quantum mechanics, as well as between Heisenberg’s and Schrödinger’s ideas. In a nutshell, Heisenberg’s matrices were to be regarded as linear operators on some vector space, whose elements were Schrödinger’s wavefunctions. The inner product that defines a Hilbert space ultimately yields all probabilities characteristic of quantum mechanics. (At a heuristic level, similar ideas had been forwarded by the physicist Dirac [7].) In honour of his mentor, the specific topological vector spaces needed in quantum mechanics were called *Hilbert spaces* by von

Neumann, who published his work on quantum mechanics in 1932 [13].

Inspired by this development, Weyl (another pupil of Hilbert’s) saw that Hilbert spaces formed an ideal setting for the theory of group representations, which turned out to play a crucial role in studying symmetries of quantum systems. The ensuing combination Hilbert space–quantum mechanics–group representations (and also ergodic theory) led von Neumann to the theory of *operator algebras* on Hilbert spaces (written down in a series of papers published between 1936 and 1949, partly with his assistant F.J. Murray). Such algebras — currently known as *von Neumann algebras* — generalize the addition and multiplication of complex matrices to infinite dimension and turn out to have an amazingly rich structure. An important extension of the class of operator algebras defined by von Neumann was introduced by Gelfand and Naimark in 1943 under the name of *C\*-algebras*; the three books of Takesaki [15] present an exhaustive survey.

**Spin geometry**

Spin geometry is a refinement of Riemannian geometry, a subject created by Bernhard Riemann in his (meanwhile) legendary Habilitation lecture ‘Über die Hypothesen, welche der Geometrie zu Grunde liegen’, delivered in Göttingen in 1854. In this lecture, Riemann proposed a vast generalization of the non-Euclidean geometries that had independently been discovered earlier that century by Gauss (unpublished), Bolyai, and Lobachevsky. Riemann’s concept of geometry was based on an infinitesimal version of Pythagoras’s Theorem, so as to provide distances between points. Since “ $a^2 = b^2 + c^2$ ”, this had the consequence that geometric quantities tend to be *quadratic* in coordinates and/or derivatives. For example, for any Riemannian geometry the Laplacian  $\Delta = \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_n^2}$  may be intrinsically defined as a second-order

partial differential operator.

Through the Laplacian, the Schrödinger equation for the quantum-mechanical wavefunction contains second-order derivatives for the spatial coordinates, but it only involves a first-order derivative in time. This unequal treatment of space and time coordinates bothered various physicists in the late 1920’s, because it precludes consistency with Einstein’s theory of (special) relativity. Through sheer guesswork, in 1928 Paul Dirac found an equation (now named after him) that is *first-order* in all coordinates, at the price of extending (scalar) wave-functions to four-component *spinors*.

Dirac’s equation turned out to have sensational consequences in physics (like the prediction of antimatter), but it also interested mathematicians. Through contributions by Hermann Weyl, Charles Ehresmann

and others, this interest eventually led to the discovery of a special class of Riemannian manifolds called *spin manifolds*, which admit geometric quantities that are *first-order* in the coordinates and/or derivatives. In particular, the (generalized) Dirac equation for spinors on such manifolds was first written down in 1963 by Michael Atiyah and Isadore Singer. Atiyah and Singer were actually unaware of Dirac’s original equation; their contribution was made in the (then) purely mathematical context of *index theory*, for which they would receive the Abel Prize in 2004, cf. [10]. In any case, spin geometry, the Dirac equation, and index theory are closely related [11], and it is their combination that Connes in turn combined with the theory of operator algebras in creating noncommutative geometry.

cluster.

In order to combine operator algebras and spin geometry, Connes invented a whole arsenal of new techniques and ideas, drawing also from other areas of mathematics (like homological algebra and algebraic topology). An important feature of his work is the interplay between abstract theory and examples. The ensuing theory of ‘noncommutative geometry’ turned out to have a wide range of applications, both in mathematics (ranging through algebra, analysis, geometry, number theory, and stochastics) and physics (especially in

solid state physics and elementary particle physics).

Besides Connes’s own book [2] and his book with Marcolli [4], a good place to start is [9]. In the Netherlands, so far there have been two MRI Master Classes specifically devoted to teaching noncommutative geometry to advanced M.Sc. students: the first was in 2003–2004, and the second took place in 2009–2010. To the surprise and delight of the participants (coming from all over the world), the closing dinner of the latter was attended by Connes himself!

**Early days**

Connes’s early work, for which he was awarded the Fields Medal in 1982, was concerned with the classification of von Neumann algebras.

*This wasn’t a fashionable area at the time. One might even say that operator algebras formed a rather introverted and isolated area of mathematics. It would have been more natural for a talented young mathematician in Paris to go to Grothendieck. Weren’t you attracted by him?*

“I started research in 1970 and at that time I was actually *repelled* by the intellectual arrogance of the followers of Grothendieck. I never liked fashionable subjects and tried to find an area of mathematics as remote as possible from algebraic geometry.”

Decades later, Connes had a change of heart towards Grothendieck, both personally and mathematically. Although they never met, Connes came to appreciate Grothendieck’s personality through the latter’s autobiographical memoir *Récoltes et Semailles*, and in addition noncommutative geometry and algebraic geometry turned out to have surprising relevance to each other [4].

*Presumably Gelfand must have been one of your favourite mathematicians? Did you have much contact with him?*

“Neither. Sure, we met and discussed mathematics a couple of times. But I found him difficult to interact with. Of course, mathematically he was a big influence and quite often one step ahead. In operator algebras I

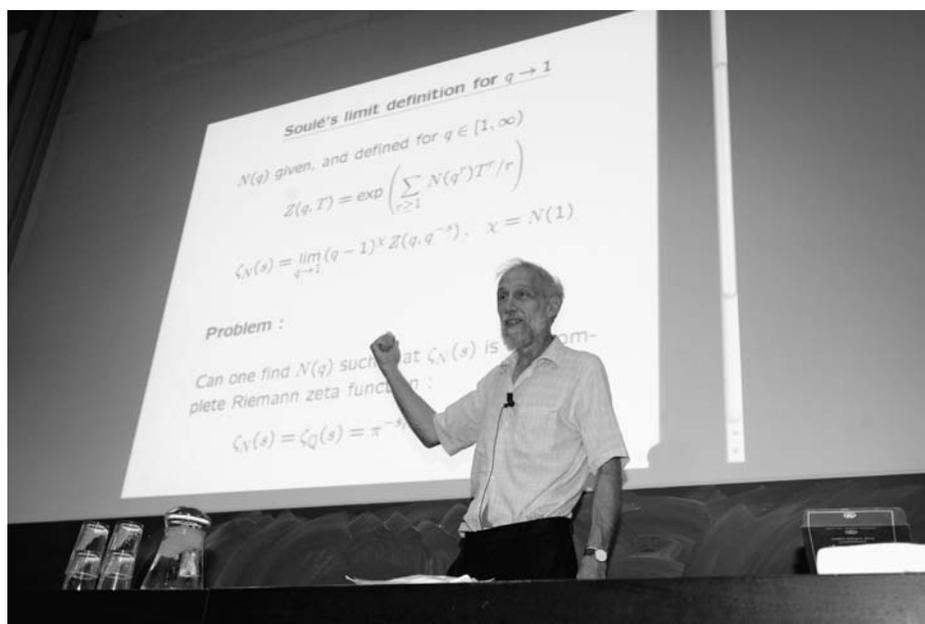


Foto: Bert Beelen

Connes during his lecture in Nijmegen, June 29, 2010

**“Time emerges from noncommutativity”**

Some familiarity with Hilbert spaces (assumed separable for simplicity) and operator theory is assumed in this box.

If  $\mathcal{M}$  is some set of bounded operators on a Hilbert space  $H$ , the *commutant*  $\mathcal{M}'$  of  $\mathcal{M}$  consists of all bounded operators on  $H$  that commute with all elements of  $\mathcal{M}$ . We say that  $\mathcal{M}$  is a *von Neumann algebra* if  $\mathcal{M}'' = \mathcal{M}$ . Indeed, such an  $\mathcal{M}$  is automatically closed under addition and multiplication of operators, with the further properties that  $A^* \in \mathcal{M}$  whenever  $A \in \mathcal{M}$  (where  $A^*$  is the Hermitian conjugate or adjoint of  $A$ ), and if  $A_n \Psi \rightarrow A\Psi$  for all  $\Psi \in H$ , and  $A_n \in \mathcal{M}$  for all  $n$ , then  $A \in \mathcal{M}$  (in other words,  $\mathcal{M}$  is strongly closed). Conversely, if  $\mathcal{M}$  satisfies all these closure properties and contains the unit operator, then  $\mathcal{M}'' = \mathcal{M}$ : this *bicommutant theorem* is the earliest result about von Neumann algebras.

A problem to which Connes made decisive contributions already in his PhD thesis is the classification of von Neumann algebras up to algebraic isomorphism. In that context, it may be assumed without loss of generality that the Hilbert space  $H$  on which  $\mathcal{M}$  acts contains a vector  $\Omega$  that is *cyclic* and *separating* for  $\mathcal{M}$ , in that  $\mathcal{M}\Omega = \{A\Omega, A \in \mathcal{M}\}$  is dense in  $H$ , and  $A\Omega = 0$  for  $A \in \mathcal{M}$  implies  $A = 0$ , respectively. (For example, if  $H = \mathbb{C}^n$ , then  $\Omega = (1, \dots, 1)$  is cyclic and separating for the algebra of all diagonal matrices.) In that situation, Tomita defined an (unbounded) antilinear operator  $S$  by  $SA\Omega = A^*\Omega$  (whose closure we denote by the same symbol). The (linear) operator  $\Delta = S^*S$  then turns out to be of great interest. Since  $\Delta$  is positive, for each  $t \in \mathbb{R}$  the operator  $\Delta^{it}$  is well defined and unitary, so that one has a ‘time-evolution’  $\sigma_t(A) = \Delta^{it}A\Delta^{-it}$  (think of  $\Delta = \exp(H)$  and  $\sigma_t(A) = A(t)$ , in which case  $\Delta^{it} = \exp(itH)$  and  $A(t) = \exp(itH)A\exp(-itH)$ , as in quantum mechanics). One of the main theorems of Tomita and Takesaki is that  $\sigma_t(A) \in \mathcal{M}$  whenever  $A \in \mathcal{M}$ . Another is that  $\sigma_t(A) = A$  for all  $A \in \mathcal{M}$  and  $t \in \mathbb{R}$  if  $\mathcal{M}$  is commutative, justifying Connes’s credo that “time emerges from noncommutativity”.

In his thesis Connes took this argument one step further by analysing the dependence of this time-evolution on  $\Omega$ . To state the simplest version of his result, assume that  $H$  contains two different vectors  $\Omega_1$  and  $\Omega_2$ , each of which is cyclic and sep-

arating for  $\mathcal{M}$ . We write  $\sigma_t^i(A)$  for the time-evolution derived from  $\Omega_i$ . Connes’s *Radon–Nikodym Theorem for von Neumann algebras* then states that there is a family  $U(t)$  of unitary operators in  $\mathcal{M}$ ,  $t \in \mathbb{R}$ , such that

$$\sigma_t^1(A) = U(t)\sigma_t^2(A)U(t)^*; \tag{1}$$

$$U(t+s) = U(s)\sigma_s^2(U(t)). \tag{2}$$

One symbolically writes  $U = D\Omega_1 : D\Omega_2$ , in terms of which one has the property  $(D\Omega_1 : D\Omega_2)^* = D\Omega_2 : D\Omega_1$ , and, in the presence of three such vectors  $\Omega_i$ , also  $(D\Omega_1 : D\Omega_2) \cdot (D\Omega_2 : D\Omega_3) = (D\Omega_1 : D\Omega_3)$ .

The proof of this theorem is based on the following idea. Extend  $\mathcal{M}$  to  $\text{Mat}_2(\mathcal{M})$ , i.e., the von Neumann algebra of  $2 \times 2$  matrices with entries in  $\mathcal{M}$ , and let  $\text{Mat}_2(\mathcal{M})$  act on  $H_2 = H \oplus H$  in the obvious way. Subsequently, let  $\text{Mat}_2(\mathcal{M})$  act on  $H_4 = H \oplus H \oplus H \oplus H = H_2 \oplus H_2$  by simply doubling the action on  $H_2$ . The vector  $(\Omega_1, 0, 0, \Omega_2) \in H_4$  is then cyclic and separating for  $\text{Mat}_2(\mathcal{M})$ , with corresponding operator  $\Delta = \text{diag}(\Delta_1, \Delta_4, \Delta_3, \Delta_2)$ . Here  $\Delta_1$  and  $\Delta_2$  are just the operators on  $H$  originally defined by  $\Omega_1$  and  $\Omega_2$ , respectively, and  $\Delta_3$  and  $\Delta_4$  are auxiliary operators on  $H$ . Denoting elements of  $\text{Mat}_2(\mathcal{M})$  by  $\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ , we obtain

$$\Delta^{it} \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{A} \end{pmatrix} \Delta^{-it} = \begin{pmatrix} \sigma_t^{(1)}(\mathbf{A}) & \mathbf{0} \\ \mathbf{0} & \sigma_t^{(2)}(\mathbf{A}) \end{pmatrix}, \tag{3}$$

with

$$\sigma_t^{(1)}(\mathbf{A}) = \begin{pmatrix} \Delta_1^{it} A_{11} \Delta_1^{-it} & \Delta_1^{it} A_{12} \Delta_4^{-it} \\ \Delta_4^{it} A_{21} \Delta_1^{-it} & \Delta_4^{it} A_{22} \Delta_4^{-it} \end{pmatrix}; \tag{4}$$

$$\sigma_t^{(2)}(\mathbf{A}) = \begin{pmatrix} \Delta_3^{it} A_{11} \Delta_3^{-it} & \Delta_3^{it} A_{12} \Delta_2^{-it} \\ \Delta_2^{it} A_{21} \Delta_3^{-it} & \Delta_2^{it} A_{22} \Delta_2^{-it} \end{pmatrix}. \tag{5}$$

But by the Tomita–Takesaki theorems, the right-hand side of (3) must be of the form  $\text{diag}(\mathbf{B}, \mathbf{B})$  for some  $\mathbf{B} \in \text{Mat}_2(\mathcal{M})$ , so that  $\sigma_t^{(1)}(\mathbf{A}) = \sigma_t^{(2)}(\mathbf{A})$ . This allows us to replace  $\Delta_4^{it} A_{22} \Delta_4^{-it}$  in (4) by  $\Delta_2^{it} A_{22} \Delta_2^{-it}$ . We then put  $U(t) = \Delta_1^{it} \Delta_4^{-it}$ , which, unlike either  $\Delta_1^{it}$  or  $\Delta_4^{-it}$ , lies in  $\mathcal{M}$ , because each entry in  $\sigma_t^{(1)}(\mathbf{A})$  must lie in  $\mathcal{M}$  if all the  $A_{ij}$

do, and here we have taken  $A_{12} = 1$ . All claims of the theorem may then be verified using elementary computations with  $2 \times 2$  matrices. For example, combining the identity

$$\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

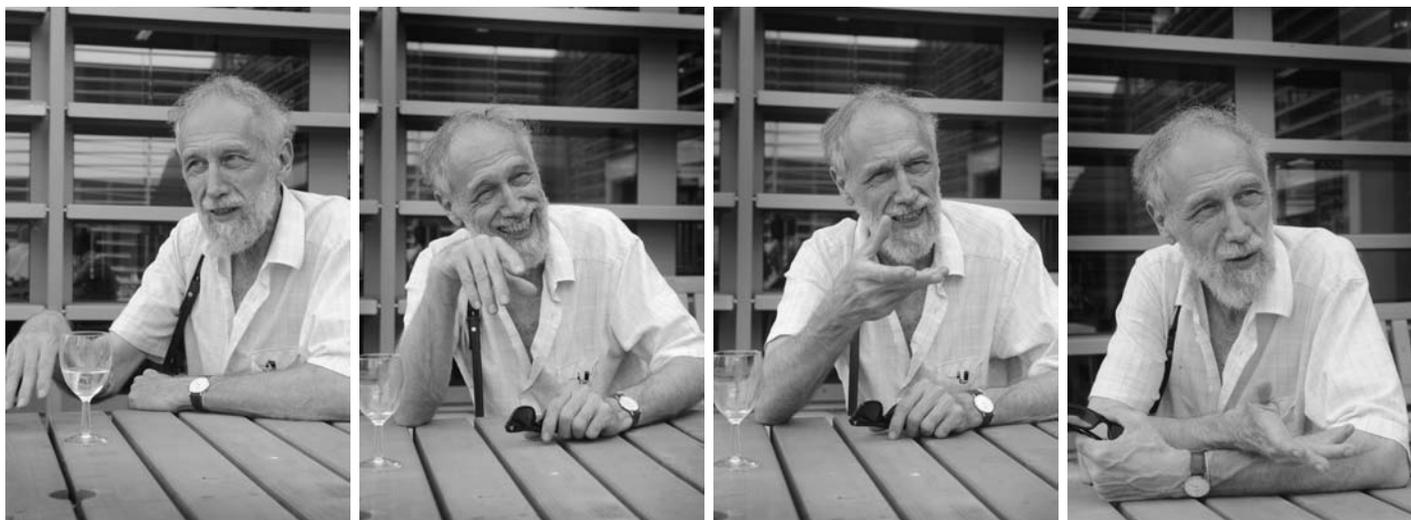
with the property  $\sigma_t^{(1)}(\mathbf{A}\mathbf{B}) = \sigma_t^{(1)}(\mathbf{A})\sigma_t^{(1)}(\mathbf{B})$ , we recover (1). Using the identity

$$\begin{pmatrix} 0 & U_t \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & U_t \end{pmatrix}$$

and evolving each side to time  $s$ , we arrive at (2).

A ‘Proof from the Book’!

Let us mention the main use of this result. An *automorphism* of  $\mathcal{M}$  is a linear map  $\sigma : \mathcal{M} \rightarrow \mathcal{M}$  satisfying  $\sigma(AB) = \sigma(A)\sigma(B)$  and  $\sigma(A^*) = \sigma(A)^*$ . The set of all automorphisms of  $\mathcal{M}$  forms a group  $\text{Aut}(\mathcal{M})$  under composition. With  $\Omega$  fixed,  $\sigma_t$  is an automorphism of  $\mathcal{M}$  for each  $t$ , and the map  $t \mapsto \sigma_t$  is a group homomorphism from  $\mathbb{R}$  (as an additive group) to  $\text{Aut}(\mathcal{M})$ . Its image  $\sigma_{\mathbb{R}}$  is a subgroup of  $\text{Aut}(\mathcal{M})$ , which depends on  $\Omega$ . However, call an automorphism  $\sigma$  of  $\mathcal{M}$  *inner* if it is of the form  $\sigma(A) = UAU^*$  for some unitary  $U \in \mathcal{M}$ . The inner automorphisms of  $\mathcal{M}$  form a normal subgroup  $\text{Inn}(\mathcal{M})$  of  $\text{Aut}(\mathcal{M})$ , with quotient group  $\text{Out}(\mathcal{M}) = \text{Aut}(\mathcal{M})/\text{Inn}(\mathcal{M})$ . Connes’s Radon–Nikodym Theorem then implies that the subgroup  $\sigma_{\mathbb{R}}/\text{Inn}(\mathcal{M})$  of  $\text{Out}(\mathcal{M})$  is independent of  $\Omega$ , and hence is an invariant of  $\mathcal{M}$ . This insight formed the basis for all subsequent progress in the classification problem (largely due to Connes himself, Haagerup, and Takesaki); see [2], Chapter v.



am too young to have met von Neumann, but I was much more influenced at a personal level by the Japanese: Tomita and also Takesaki.” Minoru Tomita (1924) is a Japanese mathematician who became deaf at the age of two and, according to Connes, had a mysterious and extremely original personality. His work on operator algebras in 1967 was subsequently refined and extended by Masamichi Takesaki and is known as Tomita–Takesaki Theory (see box on previous page). It formed the es-

sential stepping stone between the first steps in the classification of von Neumann algebras taken by Murray and von Neumann, and the work of Connes. A key ingredient of Connes’s contribution was his cocycle Radon–Nikodym Theorem for von Neumann algebras, whose proof is based on a trick with  $2 \times 2$  matrices (see box on previous page). “This remains my favourite result. It is very dear to me. Although at the end of the day the argument was very simple, it followed months of calcu-

lations and then came to me in a flash.”

**Style of working**

*In a flash! Is that the way you typically arrive at crucial insights, as suggested also by Poincaré?*

“Hardly. This happened extremely rarely, as it did with my idea that renormalization in quantum field theory corresponds to the Birkhoff decomposition (see box below). But even these so-called flashes of insight were

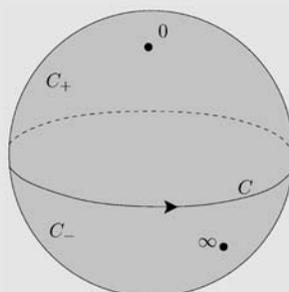
**Renormalization as a Birkhoff decomposition**

Quantum field theory was initially developed in the late 1920’s by Dirac, Heisenberg, and Pauli in order to describe electro-dynamical processes in a quantum-mechanical way. This turned out to lead to infinities in the calculations, whose systematic removal was achieved by Feynman, Schwinger, Tomonaga, and Dyson in the late 1940’s. The procedure they introduced is called *renormalization*; the typical ‘Feynman diagrams’ displaying particle interactions have remained an important tool ever since. In the early 1970’s, ’t Hooft and Veltman succeeded in extending the Feynman diagram technique and ensuing renormalization procedure to the weak and strong nuclear interactions, earning them the Nobel Prize for Physics in 1979. A fundamental idea they introduced is *dimensional regularization*, in which Feynman diagrams are computed as a function of  $z = d - 4$ , where  $d$  is the dimension of space-time. The infinities then emerge as singularities in the limit  $d \rightarrow 4$ , or  $z \rightarrow 0$ , and can be removed by subtracting the singular terms in a systematic way.

Whilst physicists simply use dimension-

al regularization and renormalization as a recipe, mathematicians continue to look for a sound mathematical basis for it. In a collaboration with Dirk Kreimer, Alain Connes found a beautiful formulation in terms of a Birkhoff decomposition.

In general, a Birkhoff decomposition of a smooth invertible  $n \times n$  matrix-valued function  $f$  on the unit sphere (regarded as the equator  $C$  of the Riemann sphere  $\mathbb{S}$ ) is a product  $f = f_- \cdot \delta \cdot f_+$ . Here  $\delta(z) = \text{diag}(z^{k_1}, \dots, z^{k_n})$  for certain integers  $k_i$ , whereas the  $f_{\pm}$  are boundary values of holomorphic functions defined on the complements  $C_{\pm}$  of  $C$  in  $\mathbb{S}$  (i.e., the northern and southern hemispheres). In particular,

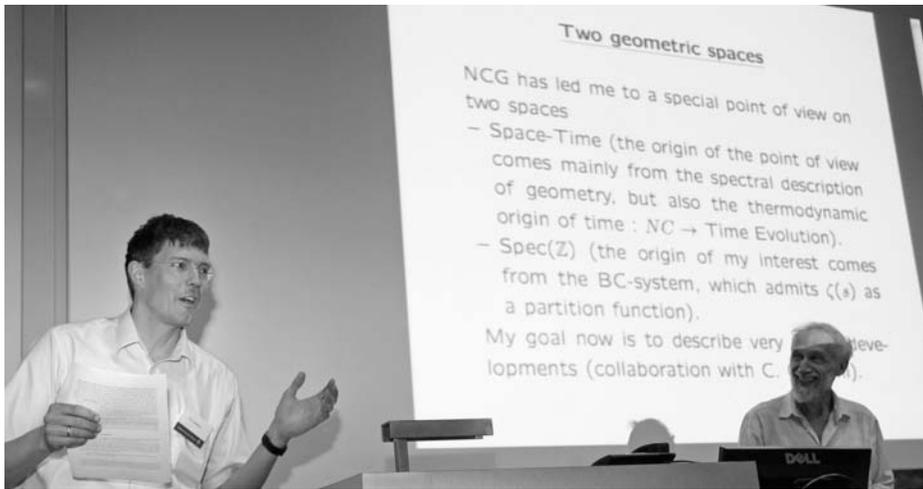


$f_+$  is finite at  $z = 0$ .

In the application to quantum field theory by Connes and Kreimer, the group of matrices in which  $f$  takes values consists of upper-triangular matrices of the form

$$\begin{bmatrix} 1 & 2 & \text{triangle diagram} \\ 0 & 1 & \text{circle diagram} \\ 0 & 0 & 1 \end{bmatrix}$$

In the first row one finds subgraphs of the graph in the upper-right corner of the matrix. Below each graph, one finds the same graph but with its subgraph(s) contracted to a vertex. Each matrix entry of  $f(z)$  is to be read as the numerical value of the corresponding Feynman diagram (as given by the so-called Feynman rules), seen as a function of  $z = d - 4$ . This forms the basis for the systematics of renormalization: the physically relevant finite part of the diagrams in question is determined by the Birkhoff decomposition of  $f$  as  $f_+(z = 0)$ .



Landsman introduces Connes's lecture in Nijmegen, June 29, 2010

the culmination of massive computations, and never came for free!"

*So what is the goal of these computations? Are they at least a path to structure, or to a theorem?*

"Long computations are, for me, a way to get into a special state of mind, into a particular mood, in which a mental picture can slowly emerge. As a preparation I go for a long walk with a particular problem in mind, and start computing in my head, before doing it in a notebook."

*We have never seen you on such a walk near Bures [i.e., Bures sur Yvette near Paris, site of the IHÉS, where Connes holds his main appointment, besides others at the College de France in Paris and Vanderbilt University in Nashville]...*

"I avoid meeting other people during such walks, especially mathematicians. I live in a remote place where I can go for a walk within a radius of 10 km around my house without meeting anyone."

*So you do all your computations in your head?*

"No, during such walks the framework of the computation is selected. Then I sit down and really start to compute. One of the dearest memories I have is a case where, with my collaborator and friend Henri Moscovici, we had to compute, separately and with a spirit of friendly competition, a cocycle which was a sum of about a thousand complicated terms. It took us three weeks of hard work, but at the end of the day a hidden Hopf algebra structure emerged from behind the scene."

Elsewhere, Connes describes the way he works in more detail [14]. To his amusement (see photo above), the following passage was read aloud by one of us as an introduction to Connes as a speaker at the GQT-conference; such open-

ness by a leading mathematician is rare:

"My impression is that I have never obtained anything at low cost. All my results have been preceded by preparatory ones, setting up work, a very long experimentation, hoping that at the end of this experimentation, an incredibly simple idea occurs which comes and solves the problem. And then you need to go through the checking period, almost intolerable because of the fear you have of being mistaken. I will never let anyone

#### Statistical mechanics and number theory

In the language of operator algebras, a quantum system is described by a  $C^*$ -algebra  $A$  (representing the physical observables) endowed with a one-parameter group  $\alpha$  of automorphisms of  $A$  (describing time evolution). In this context, the Gibbs equilibrium states of quantum statistical mechanics are described by an operator-algebraic condition first proposed by Haag, Hugenholtz, and Winnink in 1967, which replaces the classical notion of a partition function counting the energy levels  $E_i$  with the well-known temperature-dependent weights  $\exp(-E_i/k_B T)$ . The states selected by this condition are closely related to the vectors  $\Omega$  in the Tomita-Takesaki theory (see box "Time emerges from noncommutativity" on one of the previous pages); for example, the time-evolution  $\sigma_t$  induced by  $\Omega$  turns out to coincide with the physical time-evolution  $\alpha_t$ .

In the 1990's, Jean-Benoît Bost and Alain Connes discovered a quantum statistical mechanical system with two interesting number theoretical features. The first is that its partition function is the Riemann zeta function (see box on next page). The second relates to its equilibrium states: in

believe that you can wait just like this until results come all by themselves. I spent the whole summer [of 2006] checking a formula [...] There is always this permanent fear of error which doesn't improve over the years. ... And there is this part of the brain which is permanently checking, and emitting warning signals. I have had haunting fears about this. For example, some years ago, I visited Joachim Cuntz in Germany, and on the return train I looked at a somewhat bizarre example of my work with Henri Moscovici on the local index theorem. I had taken a particular value of the parameter and I convinced myself on the train that the theorem didn't work. I became a wreck — I saw that in the eyes of the people I crossed on the suburban train to go back home. I had the impression that they read such a despair in me, they wanted to help. ... Back home, I tried to eat, but I couldn't. At last, taking my courage in both hands, I went to my office and I redid the verifications. And there was a miracle which made the theorem work out in this case. ... I have had several very distressing episodes like this."

*So doing mathematics is largely a struggle... Your favourite composer must be Beethoven, rather than, say, Mozart, to whom*

the high temperature range, there is a (boring) unique equilibrium state for the system, but at a specific low temperature, there is a phase transition at which an infinite simplex of equilibrium states suddenly emerges, whose extremal points (physically corresponding to pure thermodynamic phases) are naturally indexed by... the abelianized Galois group of the rational numbers! This is an infinite topological group that plays a central role in algebraic number theory (especially in class field theory); here the relevant property is that it controls the ambiguity in distinguishing roots of the equation  $X^n - 1 = 0$  in purely algebraic terms.

This discovery was followed by a phase of intense activity, trying to use the theory as a tool for the explicit determination of the maximal abelian extension of other number fields, but up to now, all results have been reformulations of known (deep) number-theoretical descriptions in terms of quantum statistical mechanics. In this language, explicit class field theory is equivalent to the description of a suitable set of algebraic generators for the  $C^*$ -algebra of the system.

### Noncommutative geometry and number theory

How could a theory that has its roots in physics and differential geometry have anything to do with number theory? Historically, there have been various attempts at applying physical ideas to such elusive number-theoretical problems as the *Riemann hypothesis*, which says that all the zeros of the (analytic continuation of) the Riemann zeta function  $\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$  with real part between 0 and 1, actually have real part equal to 1/2. This seemingly innocuous statement is vital to many deep number theoretical results.

As was observed by Pólya already before WW1, the Riemann hypothesis would follow if the imaginary parts of the zeros of  $\zeta(s)$  would be the eigenvalues of a self-adjoint operator. But what should this operator be, or, why should it even exist? In the 1950's, Selberg observed that in the theory of Riemann surfaces, one can define a zeta function by replacing 'primes' by 'lengths of primitive geodesics' (note that shortest paths, like primes, cannot be broken up into smaller pieces), and then proved the famous *trace formula* named after him, which establishes a relation between this zeta function and the Laplace operator on

the surface. You should get excited now, because this Laplace operator is a self-adjoint unbounded operator. The main contribution of Connes from the 1990's was his use of noncommutative geometry to write down an analog of the trace formula of Selberg that is actually equivalent to the Riemann hypothesis. At this point, it was really necessary to use a *noncommutative* underlying space, since Connes's trace formula can never materialize on a usual commutative manifold.

The initial optimism that the Riemann hypothesis would now soon follow by 'just' proving the Connes trace formula from the (noncommutative) geometry of the underlying space has been converted into various high-tech long-term programs in noncommutative geometry, which increasingly seem to involve a synthesis with the algebraic and arithmetic geometry of Grothendieck and followers, and are producing interesting spin-offs—much like Kummer's theory of ideals, which failed to directly prove Fermat's Last Theorem, but produced the entire field of algebraic number theory. So let us wait and see...

Guillemin for foliations, and using a natural noncommutative space coming from the work with Bost, one obtained the Riemann–Weil explicit formulas as a trace formula and also a spectral realization of the zeros of zeta. The explicit formulas show very clearly that due to the archimedean places one needs not only an analogue of the curve used by Weil in his proof in characteristic  $p$  but also an ambient space, which in the above construction is noncommutative. But to transplant the geometric ideas of Weil, which we started doing in our collaboration with Consani and Marcolli, one needs another version of that construction, in which the points are concretely realized as valuations and the Galois ambiguity is completely respected. It is a very difficult problem but it has many interesting byproducts as shown in the recent work with Consani, which was the subject of my talk here." See also boxes on this and previous page.

*Yuri Manin maintains that a proof of the Riemann Hypothesis that does not fit into some program would not be interesting...*

"The hope is that this problem cannot be resolved without unfolding the hidden geometric structure of the above mysterious curve and its ambient space. I share this hope completely."

### Platonism

*In your book 'Triangle of thoughts' [3] you come out as an outspoken Platonist. So you think that the kind of structure we just talked about is going to be discovered, rather than invented?*

"Prime numbers are as real for me as this table. For me, mathematical reality is completely analogous to physical reality."

*But we are in Holland now, so we have to follow L.E.J. Brouwer in believing that mathematics is constructed by the human mind. Do you have a cogent argument for your Platonic position?*

"My position comes from the very important distinction between provability and truth. This is very well explained in a book by Jean-Yves Girard [8] and it would take too long to explain here, but I urge the reader to study this book. We, mathematicians, are stuck in something like a court of justice, making deductions with limited information about an external reality which I call 'archaic reality'. In this reality, there are facts, true sentences, that are not provable in the court of justice, as shown by Gödel. Gregory Chaitin even showed that almost all true statements are not provable in the court. Anyway, the point is the existence of true — not just undecidable

*his music apparently appeared from Heaven, without any effort?*

"Actually, my favourite composer is Chopin. One of my ambitions remains to play all his Preludes well, especially number 8 these days. Each one is a perfectly homogeneous world of its own and has a different sound, with an implicit idea behind. It must have been a monumental struggle to manage to express these ideas so well into written music. A fascinating aspect of music — not only Chopin's, of course — is that it allows one to develop further one's perception of the passing of time. This needs to be understood much better. Why is time passing? Or better: why do we have the impression that time passes? Is it because we are immersed in the heat bath of the 3K radiation from the big bang?"

### Riemann hypothesis

Connes's own research area is not as remote from such questions as one might think. Indeed, an idea he repeatedly expresses is that "time emerges from noncommutativity". Even thermodynamics arises from noncommutative geometry [5]: "Not only do [noncommutative algebras] generate their own

time, but they have features which enable you to cool them down or warm them up. You can do thermodynamics with them."

In the late 1960's, besides the work of Tomita and Takesaki, also the Dutch mathematical physicists Nico Hugenholtz and Marinus Winnink (in collaboration with Rudolf Haag) played an important role in relating operator algebras to time and thermodynamics. As Connes remarks, the ensuing link between Tomita–Takesaki Theory for von Neumann algebras and quantum statistical physics "has become an indisputable point of interaction between theoretical physics and pure mathematics" [2], p. 42. See also box "Time emerges from noncommutativity" and the box on the previous page.

*In your recent book with Matilde Marcolli [4] you even develop a thermodynamical approach to number theory and the Riemann Hypothesis. You seem to have taken up the highest challenge in pure mathematics. What do you expect?*

"It started with my work with Bost in the early 1990's on phase transitions on Hecke algebras. The Riemann zeta function came naturally as the partition function. Then in 1996, I showed that, using a formula due to



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— but unprovable statements. Unless you understand that point it is worthless to debate about Platonism.”

### Physics

*What is the place of physics in this archaic reality? How do you see the relationship between physics and mathematics?*

“The standard wisdom is that mathematics is the language in which physics is written and I certainly agree with that, but my hope is that the relation goes much further and that basic concepts such as the passing of time will only be really understood through the unfolding of deeper mathematics. The boundary line between physics and mathematics comes ultimately from the motivation of physicists to model reality and thus confront their predictions with experiment.”

*You actually made a physical prediction from noncommutative geometry, i.e., the mass of the Higgs particle. The result was slightly off experimental bounds. How are you going to keep physicists on board?*

“This prediction was based on the hypo-

thesis of the ‘big desert’, namely that there will be no new physics up to the unification scale, besides the Standard Model coupled to gravity. Thus it was a bit like trying to see a fly in a cup of tea by looking at the earth from another planetary system. But very strangely the model also predicted the correct mass of the top quark, and a surprising number of mechanisms such as the Higgs and the seesaw mechanisms. After our work in 1996 with Ali Chamseddine we gave up in 1998 because of the discovery of neutrino mixing, only to understand 8 years later that there was one case which we had overlooked, namely allowing the KO-theory dimension of the finite space to be 6 modulo 8, and which was giving for free the neutrino mixing and much better features for the model. [All this is explained in Connes’s book with Marcolli [4].] As it is now, I stopped doing such calculations and will just wait for the experiments. If supersymmetry is going to be found, it will be very hard to convince the physicists of the noncommutative geometry approach.”

*But there is no contradiction between*

*supersymmetry and noncommutative geometry.*

“You are right, but string theory would claim the ground even more than they are already doing now. In any case, the Standard Model [of elementary particle physics] is full of tricks. What we need is simplicity. I think that is what noncommutative geometry provides. The inverse line element is an operator. Its only invariants [under unitary transformations] are the eigenvalues. And these are eventually what is observed in Nature. The truth is that this simplicity is only a starting point and a lot more work would be needed to explore the quantum theory.”

*Do you have a preference for mathematics over physics?*

“My heart lies with both.”

At this appropriate point Connes had to leave for a dinner appointment with Sir Michael Atiyah, another speaker at the GQT-conference and a comparable source of inspiration for our cluster. ↵

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