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In Memoriam Hans Duistermaat (1942–2010)

Classical mechanics

One of Hans Duistermaat's favorite subjects was classical mechanics. His interest in this area was wide ranging. In this article Richard Cushman gives an overview of the papers that Duistermaat published on classical mechanics.

In order to describe what Hans did in classical mechanics, I will organize his papers in this area, somewhat arbitrarily, into three classes: periodic solutions near an equilibrium point, monodromy in integrable systems, and other topics.

Periodic solutions near an equilibrium point

In [1] Hans studied the persistence of periodic solutions near an equilibrium point of a two degree of freedom Hamiltonian system which is in $1 : 2$ resonance. This is the simplest situation where a well-known theorem of Lyapunov on the persistence of periodic solutions fails. Years later Hans returned to this subject in the almost forgotten paper [4]. Here, using the theory of singularities of mappings which are invariant under a circle action that fixes the origin, he proved a stability result for the set of short period periodic orbits near an equilibrium point of a resonant Hamiltonian system of two degrees of freedom. In particular, he showed that this set of periodic orbits is diffeomorphic to the set of critical points of rank one of the energy momentum mapping. Here the energy is the Hamiltonian of the Birkhoff normal form of the original resonant Hamiltonian truncated at some finite order. The momentum of the circle action is the quadratic term of this normal form. As far as I am concerned, this result is the definitive generalization of Lyapunov's theorem.

Monodromy in integrable systems

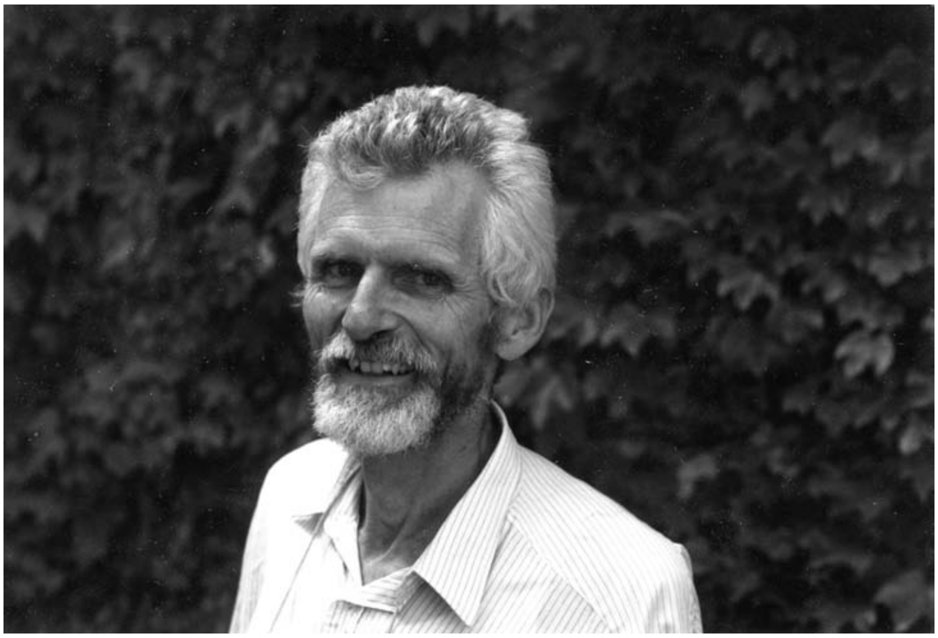
Hans's most important contribution to the geometric study of Hamiltonian systems is his discovery of the phenomenon of monodromy in [3]. To describe what monodromy is we look at a two degree of freedom Hamiltonian system on a four-dimensional phase space, which we assume is a Euclidean space. We suppose that this Hamiltonian system has another function, which is an integral, that is, it is constant on the motions of the original Hamiltonian system. Such a Hamiltonian system is said to be completely integrable with integral map given by assigning to each point in phase space the value of the Hamiltonian and the extra integral. If we assume that the integral map is proper and each preimage of a point is connected, then the action-angle theorem shows that the preimage of a suitably small open 2-disk in the set of regular values of the integral map is symplectically diffeomorphic to a product of a 2-torus and 2-disk. Hans showed that this local theorem need not hold globally. In particular, if we have a smooth closed, nonintersecting curve in the set of regular values of the integral map, then the preimage of this curve in phase space under the integral map is the total space of a 2-torus bundle, which need not be diffeomorphic to a product of the closed curve and a 2-torus. To understand what this global twisting is, we note that a 2-torus bundle over a circle may be looked at as a product bundle over a closed interval with a typical fiber of a 2-torus. Here each of its two end 2-tori, which are Euclidean 2-space modulo the lattice of points with integer coordinates, are glued together by an integer 2×2 matrix with determinant 1. The monodromy of this 2-torus bundle is just this integer matrix. If the monodromy is not the identity matrix, then the 2-torus bundle is not a product bundle. In [3] Hans gave a list of geometric and analytic obstructions for local action angle coordinates to be global. Monodromy is just the simplest obstruction. Monodromy would not be interesting if there were no two degree of freedom integrable Hamiltonian systems having it. When Hans was starting to write [3], he asked me to find an example of such a system. The next day I told him that the spherical pendulum, which was studied by Christiaan Huygens in 1612, has monodromy and gave him a proof. When writing up the paper Hans found a much simpler geometric argument to show that the spherical pendulum had monodromy. In the early days, showing that a particular integrable system had monodromy was not easy. In [8] Hans did this for the Hamiltonian Hopf bifurcation. In [6] Hans and I discovered that monodromy appears in the joint spectrum of the energy and angular momentum operators of the quantized spherical pendulum. This discovery has now been recognized as fundamental by chemists who study the spectra of molecules and has lead to a very active area of scientific research.

Nonholonomically constrained systems

In the middle 1990's Hans became interested in nonholonomically constrained systems such as the disk or a dynamically symmetric sphere with its center of mass not at its geometric center. Both are assumed to be rolling without slipping on a horizontal plane under the influence of a constant vertical gravitational force. This interest gave rise to [9]. In this paper Hans gave a simple geometric criteria for a not necessarily Hamiltonian system to have monodromy. He showed that an oblate ellipsoid of revolution rolling without slipping on a horizontal plane under the influence of a constant vertical gravitational force had a cycle of heteroclinic hyperbolic equilibria whose local monodromies added up to the identity. This shows that it cannot be made into a Hamiltonian system. The book [11] clearly indicates Hans's contributions to the geometric study of nonholonomically constrained systems. Especially, it contains a complete qualitative study of the motion of the rolling disk, some of which was published in [10].

Other topics

His remaining publications range from removing the incompleteness of the flow of the



Hans Duistermaat in 1994

Foto: Archives of the MFO

Kepler problem for all negative energies at the same time, see [7], to showing that the $1:1:2$ resonance is nonintegrable by looking at its fourth-order normal form. In two degrees of freedom, integrability cannot be decided by any finite-order normal form. In the remaining paper on periodic linear Hamiltonian systems [2] Hans answered an old question of Bott about the Morse index of iterates of a periodic geodesic. Bott showed that this index is the sum of the index of the periodic geodesic and invariants of the real symplectic conjugacy class of the linear Poincaré map. Hans gave an explicit formula for the Morse index.

My memories

Working with Hans and collaborating on our joint publications is at the core of my mathematical career. It is hard for me to realize that he cannot answer my questions any more.

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