Problemen

| Problem Section

This Problem Section is open to everyone. For each problem the most elegant solution will be rewarded with a 20 Euro book token. The problems and results can also be found on the Problem Section website www.nieuwarchief.nl/ps.
Now and then there will be a Star Problem, of which the editors do not know any solution. Whoever first sends in a correct solution within one year will receive a prize of 100 Euro.
Both suggestions for problems and solutions can be sent to uwc@nieuwarchief.nl or to the address given below in the left-hand corner; submission by email (in LaTeX) is preferred. When proposing a problem, please include a complete solution, relevant references, etc. Group contributions are welcome. Participants should repeat their name, address, university and year of study if applicable at the beginning of each problem/solution. If you discover a problem has already been solved in the literature, please let us know. The submission deadline for this edition is June 1, 2007.
The prizes for the Problem Section are sponsored by *Optiver Derivatives Trading*.

Problem A (Proposed by John Scholes)
Define the sequence $\{u_n\}$ by $u_1 = 1$, $u_{n+1} = 1 + (n/u_n)$. Prove or disprove that

$$u_n - 1 \ < \sqrt{n} \le u_n.$$

Problem B (Folklore)
Given a non-degenerate tetrahedron (whose vertices do not all lie in the same plane), which conditions have to be satisfied in order that the altitudes intersect at one point?

Problem C (Proposed by Jan Draisma)
Let e be a positive integer, and let d be an element of $\{0, 1, 2, \ldots, 3e\}$. Show that the polynomial

$$P = \sum_{a \ge 0, b \ge 0, c \ge 0, a+b+c=d} \frac{d!}{a!b!c!} \binom{e}{a}\binom{e}{b}\binom{e}{c} x^a y^b z^c$$

in the three variables x, y, and z is not divisible by $x + y + z$ unless $d = 1$.

Problem * (Proposed by Farideh Firoozbakht)
For a positive integer n, $\sigma(n)$ is defined as the sum of the divisors of n, including n, and $\phi(n)$ is the Euler phi (or totient) function. Let $\psi(n) = \sigma(n)/\phi(n)$. For $n = 2, 3, 6, 12, 14$ and 15, $\psi(n)$ is an integer (respectively $3, 2, 6, 7, 4$, and 3). Does there exist, for every integer k, an integer n such that $\psi(n) = k$.
As a reference, see: http://www.research.att.com/~njas/sequences/A055234

Edition 2006/3
For Edition 2006/3 we received submissions from N. Hekster, Jinbi Jin, C.Jonkers, R.A. Kortram, and Peter Vandendriessche. The solution of problem 2006/3-C will be published in the June issue.

Problem 2006/3-A Evaluate

$$\int_0^1 \frac{\log(x+1)}{x^2+1} \mathrm{d}x.$$

Solution This problem was solved by Jinbi Jin, C.Jonkers, R.A. Kortram, and Peter Vandendriessche. The solution below is based on that of Peter Vandendriessche.

Eindredactie: Matthijs Coster
Redactieadres: Problemenrubriek NAW
Mathematisch Instituut
Postbus 9512
2300 RA Leiden
uwc@nieuwarchief.nl

Substitute $x = \tan(t)$, then the integral can be written as

$$I = \int_0^1 \frac{\log(x+1)}{x^2+1}\mathrm{d}x = \int_0^{\frac{\pi}{4}} \log(1+\tan t)\mathrm{d}t.$$

Substitute $t = \frac{\pi}{4} - s$, then we get

$$I = \int_{\frac{\pi}{4}}^0 \log\left(1 + \frac{1-\tan s}{1+\tan s}\right)(-\mathrm{d}s) = \int_0^{\frac{\pi}{4}} \log\left(\frac{2}{1+\tan s}\right)\mathrm{d}s = \frac{\pi}{4}\log 2 - I,$$

hence $I = \frac{\pi \log(2)}{8}$.
C. Jonkers remarks that this exercise was also published in work of F. Frenet as well as in *Aufgabensammlung zur höheren Mathematik* of N. M. Günter and R. O. Kusmin, (ISBN 978-3-8171-1641-6).

Problem 2006/3-B

1. Find an integer n (as small as possible) and rational numbers $a_1, \ldots, a_n$ such that

$$\sqrt{10 + 3\sqrt{11}} = \sum_{i=1}^n \sqrt{a_i}.$$

2. Find an integer n (as small as possible) and rational numbers $b_1, \ldots, b_n$ such that

$$\sqrt[3]{\sqrt[3]{2} - 1} = \sum_{i=1}^n \sqrt[3]{b_i}.$$

Solution This problem was solved by R.A. Kortram and N. Hekster. The solution below is based on that of R.A. Kortram.
1. It is a simple matter to verify the identity $\sqrt{10+3\sqrt{11}} = \sqrt{\frac{9}{2}} + \sqrt{\frac{11}{2}}$. It is also clear that there is no rational number a such that $\sqrt{10+3\sqrt{11}} = \sqrt{a}$, because the existence of such a number a would imply that $10 + 3\sqrt{11} = a$, i.e. $\sqrt{11} \in \mathbf{Q}$; clearly a contradiction. Thus, the smallest possible integer n is 2.
2. From the trivial identity $(\sqrt[3]{2}+1)^3(\sqrt[3]{2}-1) = 3$, we derive that

$$(\sqrt[3]{2}+1)\sqrt[3]{(\sqrt[3]{2}-1)} = \sqrt[3]{3},$$

and after multiplying both sides by $\sqrt[3]{4} - \sqrt[3]{2} + 1$, we obtain

$$3\sqrt[3]{\sqrt[3]{2}-1} = \sqrt[3]{3}(\sqrt[3]{4} - \sqrt[3]{2} + 1).$$

Thus, $\sqrt[3]{\sqrt[3]{2}-1} = \sqrt[3]{\frac{1}{9}} + \sqrt[3]{-\frac{2}{9}} + \sqrt[3]{\frac{4}{9}}$.
Now assume that there exist rational numbers a and b such that

$$\sqrt[3]{\sqrt[3]{2}-1} = \sqrt[3]{a} + \sqrt[3]{b}.$$

Raising both sides of this identity to the third power leads to

$$\sqrt[3]{2} - 1 = a + b + 3\sqrt[3]{ab}(\sqrt[3]{a} + \sqrt[3]{b}) = a + b + 3\sqrt[3]{ab}(\sqrt[3]{\sqrt[3]{2}-1}).$$

This implies that $\sqrt[3]{2} - (a+b+1) = 3\sqrt[3]{ab}(\sqrt[3]{\sqrt[3]{2}-1})$. Raising both sides to the third power shows that $\sqrt[3]{2}$ satisfies an equation of degree 2 over $\mathbf{Q}$:

$$3(a+b+1)(\sqrt[3]{2})2 - (3(a+b+1) - 27ab)\sqrt[3]{2} - 2 - 27ab + (a+b+1)3 = 0.$$

Since $\sqrt[3]{2}$ has degree 3, all coefficients must be zero, i.e.

$$(a+b+1) = 0 \quad 3(a+b+1) - 27ab = 0 \quad (a+b+1)3 - 27ab - 2 = 0,$$

which is clearly impossible.