Problem/UWC

Universitaire Wiskunde Competitie

The Universitaire Wiskunde Competitie (UWC) is a ladder competition for students. Others may participate 'hors concours'. The results can also be found on internet at: http://www.nieuwarchief.nl/uwc

This issue contains four problems A, B, C, and D. A total of 12 points can be obtained for each problem: 8 for a complete and correct answer, at most 2 points for elegance, and at most 2 points for possible generalisations. To compute the overall score, the totals for each problem are multiplied by a factor 3, 4, 5, and 5, respectively.
The three best contributions will be honoured with a Sessions Prize of respectively 100, 50 and 25 Euro. The points of the winners will be added to their total after multiplication by a factor of respectively 0, 1/2 and 3/4. The highest ranked participant will be given a prize of 100 Euro, after which he/she starts over at the bottom of the ladder with 0 points.
Twice a year there is a Star Problem, of which the editors do not know any solution. Whoever first sends in a correct solution within one year will also receive a prize of 100 Euro.
Group contributions are welcome. Submission by email (in LaTeX) is preferred; participants should repeat their name, address, university and year of study at the beginning of each problem/solution. The submission deadline for this session is June 1, 2006.
The Universitaire Wiskunde Competitie is sponsored by Optiver Derivatives Trading, and the Vereniging voor Studie- en Studentenbelangen in Delft.

Rectification
In Session 2005/4 of the UWC, the names of the proposers of the resolved problems and those of the new problems were inverted.
The problems of Session 2005/2 placed with solutions were proposed by: Unknown (2005/2-A), Matthijs Coster (2005/2-B,D), and Roger Hendrickx and Rob van der Waall (2005/2-C). The new problems were proposed by: Matthijs Coster (2005/4-A,D), Jan van de Lune (2005/4-B), and Leen Bleijenga (2005/4-C).
In addition to the 5 solutions mentioned for problem 2005/2-C, we also received a very detailed solution by the two proposers, with a reference to the Ph.D. thesis of E. Barbette (Paris, 1910); p. 23 of this thesis contains a theorem equivalent to parts 1 and 2 of the proposed problem.

Problem A (Proposed by Jurjen Bos)
We are given a lamp and a sufficiently large number of synchronised time switches that can be turned on or off by the quarter of an hour and have a revolution time of 24 hours. We are going to mount a finite number of switches on top of each other, and put the lamp on top of the result. At the beginning, all time switches are synchronised at 24:00 hours. We define a *period* to be a time span in which the lamp is on for at least one quarter of an hour, and is off for at least one quarter of an hour, and which repeats itself. Which periods, shorter than 4 days, can be constructed?

Problem B (Proposed by Matthijs Coster)
Let $P = (0,0), Q = (3,4)$. Find all points $T = (x, y)$ such that
- x and y are integers,
- the lengths of line segments PT and QT are integers.

Problem C (Proposed by Johan Bosman)
Let $n \geq 1$ be an integer and $f(x) = a_n x^n + \cdots + a_0$ be a polynomial with real coefficients. Suppose that f satisfies the following condition:

$$|f(\xi)| \leq 1 \quad \text{for each } \xi \in [-1,1].$$

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Oplossingen

Consider the polynomial

$$g(x) = a_0x^n + \cdots + a_n,$$

the reciprocal polynomial of f. Show that g satisfies

$$|g(\xi)| \leq 2^{n-1} \quad \text{for each } \xi \in [-1, 1].$$

Problem D (Proposed by Michiel Vermeulen)
This problem was proposed as Problem 2005/3-B(2). Since no solutions were submitted, the editors of the problem section have decided to reformulate this part.
Let G be a group such that the maps $f_m, f_n : G \to G$ given by $f_m(x) = x^m$ and $f_n(x) = x^n$ are both homomorphisms.

- Show that G is Abelian if (m, n) is one of the pairs (4,11), (6,17).
- Show that there are infinitely many pairs (m, n) such that G is Abelian.
- Show that for every m there are infinitely many n such that G is Abelian.
- Given a pair (m, n), how are we able to predict whether G is Abelian?

Edition 2005/3
For Session 2005/3 of the Universitaire Wiskunde Competitie we received submissions from DESDA (Nijmegen), Ruud Jeurissen, the team A.P.M. Kupers en J.W.T. Konter, and Jaap Spies.

Problem 2005/3-A In what follows f, g are two continuous functions.
1) Determine $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ such that $f \circ g(x) = x + 1$ and $g \circ f(x) = x - 1$.
2) Determine $f : \mathbf{R}^+ \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ such that $f \circ g(x) = x + 1$ and $g \circ f(x) = 2x$.
As usual, the symbol '$\circ$' denotes the composition of functions and $\mathbf{R}^+$ the set of all strict positive real numbers.

Solution This problem was solved by DESDA (Nijmegen), Ruud Jeurissen and the team A.P.M. Kupers en J.W.T. Konter. The solution below is based on that of Ruud Jeurissen.
1) $f(x) + 1 = f \circ g \circ f(x) = f(x - 1)$, so there is an a such that $f(x) = -x + a$. $g(x) - 1 = g \circ f \circ g(x) = g(x + 1)$, so there is a b such that $g(x) = -x + b$. Then $f \circ g(x) = f(-x + b) = x - b + a$, so we must have $a - b = 1$, in which case $g \circ f(x) = g(-x + a) = x - a + b = x - 1$, as desired.
2) Since $f \circ g$ is defined, g can only take positive values. For $x > 0$ we have $f(x) + 1 = f \circ g \circ f(x) = f(2x)$, so there is a b such that $f(x) = {}^2\log x + b$. For all x we have $2g(x) = g \circ f \circ g(x) = g(x + 1)$, so there is an a such that $g(x) = 2^{x+a}$. Then $f \circ g(x) = f(2^{x+a}) = x + a + b$, so we must have $a + b = 1$, in which case $g \circ f(x) = g({}^2\log x + b) = x \cdot 2^{b+a} = 2x$, as desired.

Problem Generalisation The team A.P.M. Kupers en J.W.T. Konter considered the following generalisation of part 2):
Determine f and g such that $f \circ g(x) = x + 1$ and $g \circ f(x) = ax + b$. They found that f and g must satisfy

$$f(x) = \frac{Log\left(\frac{(ax+b)(a-1)+b}{b+ca-c}\right)}{Log(a)}, \qquad g(k) = \frac{(a^k - 1)b}{a - 1} + c \cdot a^k$$

Problem 2005/3-B

1. Let G be a group and suppose that the maps $f, g : G \to G$ with $f(x) = x^3$ and $g(x) = x^5$ are both homomorphisms. Show that G is Abelian.
2. In the previous exercise, by which pairs (m, n) can $(3, 5)$ be replaced if we still want to be able to prove that G is Abelian.

Oplossingen

Solution This problem was solved by Jaap Spies and the team A.P.M. Kupers en J.W.T. Konter. The solution below is based on that of Jaap Spies.
By assumption we have $(ab)^5 = a^5b^5$ for all $a, b \in G$. We easily see that $(ba)^4 = a^4b^4$. Likewise, $(ab)^3 = a^3b^3$ for all $a, b \in G$ and hence $(ba)^2 = a^2b^2$. So $(a^2b^2)^2 = a^4b^4$ and $b^2a^2 = a^2b^2$. Hence squares commute in G. Now $a^4b^4 = b^4a^4 = (ba)^4$ and so $b^3a^3 = (ab)^3 = a^3b^3$. Hence cubes also commute in G. In the solution of Problem 2003/4-B of the UWC, it was proved that in this case G is Abelian.

Problem 2005/3-C For $s > 1$ define

$$\psi_1(s) = \prod_p (1 - \frac{1}{p^s})^{-1} \text{ with } p \text{ over all primes } \equiv 1 \bmod 4$$

and

$$\psi_3(s) = \prod_q (1 - \frac{1}{q^s})^{-1} \text{ with } q \text{ over all primes } \equiv 3 \bmod 4.$$

Describe how $\lim_{s\downarrow 1} \frac{\psi_3(s)}{\psi_1(s)}$ can be computed to 'any' degree of (high) accuracy (precision). (The use of an algebra-package is permitted.)

Solution This problem has been solved by the team A.P.M. Kupers en J.W.T. Konter. The solution below is based on their solution. It has been shortened for publication; the complete text, with references and calculations, can be found on the UWC website.

The relation between ψ_1, ψ_3 and the ζ-function
The Riemann-Zeta function is given by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

It can also be written as a product over the prime numbers:

$$\zeta(s) = \prod_{n=1}^{\infty} \left(1 - \frac{1}{p_n^s}\right)^{-1}$$

where p_n is the n-th prime number. As all odd prime numbers are either 1 or 3 modulo 4, we can rewrite this as:

$$\zeta(s) = \frac{1}{\left(1 - \frac{1}{2^s}\right)} \psi_1(s)\psi_3(s).$$

Is the ratio $\psi_3(1)/\psi_1(1)$ a real number?
Both $\psi_3(1)$ and $\psi_1(1)$ are infinite. Is the ratio $\psi_3(1)/\psi_1(1)$ a real number? Using the number theoretic character χ_4, we can prove that this ratio is equal to $(4/\pi)\psi_3(2)$, which is indeed a real number.

The ratio $\psi_1(1)/\psi_3(1)$
Likewise, we can show that $\psi_1(1)/\psi_3(1)$ is equal to $(2/\pi)\psi_1(2)$.

The idea behind the approximation
If we divide the expression that we found for $\psi_3(1)/\psi_1(1)$ by the one we found for $\psi_1(1)/\psi_3(1)$, we obtain

$$\left(\frac{\psi_3(1)}{\psi_1(1)}\right)^2 = 2\frac{\psi_3(2)}{\psi_1(2)}, \qquad \frac{\psi_3(1)}{\psi_1(1)} = \sqrt{2\frac{\psi_3(2)}{\psi_1(2)}}.$$

The idea behind the approximation is that $\psi_3(2)/\psi_1(2)$ can again be written as the square root of a constant times $\psi_3(4)/\psi_1(4)$, etc. This turns out to be correct.

The Dirichlet L-Series
The Dirichlet L-series is defined as

$$L_k(s,\chi_k) = \sum_{n=1}^{\infty} \frac{\chi_k(n)}{n^s}.$$

We can show that

$$L_4(s,\chi_4) = \prod_n \left(1 - \frac{\chi_4(p_n)}{p_n^s}\right)^{-1}$$

Now L_4 is equal to the Dirichlet β-function, that is,

$$L_4(s,\chi_4) = \beta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}$$

In particular, the sum in the β-function is over all integers, and not just the primes.

The ratios $\frac{\psi_3(2^n)}{\psi_1(2^n)}$ and $\frac{\psi_1(2^n)}{\psi_3(2^n)}$
If we divide the ratio $\psi_3(2^n)/\psi_1(2^n)$ for n$\in \mathbf{N} \cup \{0\}$ by $\psi_3(2^{n+1})$, then through calculations as above, we obtain

$$\frac{\psi_3(2^n)}{\psi_3(2^{n+1})\psi_1(2^n)} = \prod_p \left(1 - \frac{1}{p^{2^n}}\right) \prod_q \left(1 + \frac{1}{q^{2^n}}\right).$$

The result is equal to $[L_4(2^n,\chi)]^{-1}$, which in turn is equal to $\beta(2^n)^{-1}$. For $\psi_3(2^n)/\psi_1(2^n)$ we therefore find the expression $\psi_3(2^{n+1})/\beta(2^n)$. Let us now consider $\psi_1(2^n)/\psi_3(2^n)$:

$$\frac{\psi_1(2^n)}{\psi_1(2^{n+1})\,\psi_3(2^n)} = \prod_q \left(1 - \frac{1}{q^{2^n}}\right) \prod_p \left(1 + \frac{1}{p^{2^n}}\right).$$

We can again recognise the Dirichlet L-series in here, and after some calculation, we find

$$\frac{\psi_1(2^n)}{\psi_3(2^n)} = \frac{\beta(2^n)}{\left(1 - \frac{1}{2^{2^{n+1}}}\right)\zeta(2^{n+1})} \psi_1\left(2^{n+1}\right)$$

A recursive formula
If we now divide the expression we found for $\psi_3(2^n)/\psi_1(2^n)$ by the one we found for $\psi_1(2^n)/\psi_3(2^n)$, we find

$$\left(\frac{\psi_3(2^n)}{\psi_1(2^n)}\right)^2 = \frac{\left(1 - \frac{1}{2^{2^{n+1}}}\right)\zeta(2^{n+1})}{\beta(2^n)^2} \frac{\psi_3(2^{n+1})}{\psi_1(2^{n+1})}$$

$$\frac{\psi_3(2^n)}{\psi_1(2^n)} = \sqrt{\frac{\left(1 - \frac{1}{2^{2^{n+1}}}\right)\zeta(2^{n+1})}{\beta(2^n)^2} \frac{\psi_3(2^{n+1})}{\psi_1(2^{n+1})}}.$$

This is recursive formula that allows us to deduce the ratio $\psi_3(2^n)/\psi_1(2^n)$ from the ratio $\psi_3(2^{n+1})/\psi_1(2^{n+1})$. Each time this recursive formula is applied to a ratio $\psi_3(s)/\psi_1(s)$, s is divided by 2. This way we can approximate the ratio $\psi_3(1)/\psi_1(1)$. We do not need to use sums or products over primes because both ζ and β can be approximated without these, for example using an algebra-package such as *Mathematica* or *Matlab*. Consider the following limits:

$$\lim_{s\to\infty} \psi_1(s) = \lim_{s\to\infty} \prod_p \frac{1}{\left(1 - \frac{1}{p^s}\right)} = 1, \qquad \lim_{s\to\infty} \psi_3(s) = \lim_{s\to\infty} \prod_q \frac{1}{\left(1 - \frac{1}{q^s}\right)} = 1$$

Consequently the following limit also tends to 1:

$$\lim_{s\to\infty} \frac{\psi_3(s)}{\psi_1(s)} = 1$$

We can therefore approximate $\psi_3(1)/\psi_1(1)$ as follows:

- Choose a positive integer n.
- Approximate $\frac{\psi_3(2^n)}{\psi_1(2^n)}$ by supposing that $\frac{\psi_3(2^n)}{\psi_1(2^n)} = 1$.
- Use the recursive formula a number of times to obtain $\frac{\psi_3(1)}{\psi_1(1)}$.

The ratio $\psi_3(1)/\psi_3(1)$ can thus be approximated by the following limit:

$$\lim_{n\to\infty} \sqrt{\frac{\left(1-\frac{1}{2^2}\right)\zeta(2)}{\beta(1)^2}\sqrt{\frac{\left(1-\frac{1}{2^4}\right)\zeta(4)}{\beta(2)^2}\sqrt{\frac{\left(1-\frac{1}{2^8}\right)\zeta(8)}{\beta(4)^2}\cdots\sqrt{\frac{\left(1-\frac{1}{2^{2*2^n}}\right)\zeta\left(2^{n+1}\right)}{\beta\left(2^n\right)^2}*1}}}}$$

The precision of the approximation depends on three factors:

- The number n: the larger n is, the better the approximation.
- The precision used in the approximation of the ζ and β-functions: the more precise these are, the better the approximation of the ratio. Nowadays, with algebra-packages such as *Mathematica* and *Maple*, this is no problem.
- The precision used in calculation the square root: don't forget that the square root is also an approximation. *Mathematica* and *Maple* have no problem with this.

A trial approximation with Mathematica

The following functions approximate $\psi_3(s)$ and $\psi_1(s)$ by only considering the first n primes.

```
p1[s\_, n\_] :=
 Module[{x = 1}, {pmo1 =
  DeleteCases[
   Table[If[Mod[Prime[i], 4] == 1, Prime[i], 0],
                  {i, 1, n}], 0];
   Product[(1 - pmo1[[i]]$\hat{}$(-s))$\hat{}$-1,
                  {i, 1, Length[pmo1]}]}][[1]]
```

```
p3[s\_, n\_] :=
 Module[{x = 1}, {pmo3 =
  DeleteCases[
   Table[If[Mod[Prime[i], 4] == 3, Prime[i], 0],
                  {i, 1, n}], 0];
   Product[(1 - pmo3[[i]]$\hat{}$(-s))$\hat{}$-1,
                  {i, 1, Length[pmo3]}]}][[1]]
```

Let us consider the ratio for the first 10000 primes. Timing[] determines the time it takes *Mathematica* to compute this.

```
p3p1 = Timing[N[p3[1, 100000]/p1[1, 100000], 20]]
37.594 Second, 1.4871655814206811459
```

Mathematica takes about 37,5 seconds to do this.
The β-function is a sum, as is the ζ-function, but it is not standard in *Mathematica*. We must first define it:

```
DB[x\_, k\_] := Sum[(-1)$\hat{}$n/(2n + 1)$\hat{}$(x), {n, 0, k}]
```

```
BLIM[n\_] :=
 Module[{x = n, expr = 1},
  While[x != -1,
   expr = Sqrt[(1 - 2$\hat{}$(-2$\hat{}$(x + 1)))*
   Zeta[2$\hat{}$(x + 1)]/(DB[2$\hat{}$x,
      $\setminus$[Infinity]])$\hat{}$2*expr]; x = x - 1]; expr]
```

Let us first make a table with the approximations for n = $\{1, 2, 3, 4, 5\}$, then with the differences between the approximations and the value computed above with 10000 primes.

```
approximation = Table[Timing[N[BLIM[i], 20]], {i, 1, 5}]
```

```
{{0.032 Second, 1.4830557664224863250}, {0.046 Second,
 1.4872121328716638225}, {0.094 Second,
```

```
 1.4872400244256083148}, {0.125 Second,
 1.4872400265843418256}, {0.188 Second, 1.4872400265843418507}}
```

Even the best approximation only takes 0,2 seconds. But how large is the deviation?

```
Table[approximation[[i]][[2]] - p3p1, {i, 1, 5}]
```

```
-0.0041098149981948210, 0.0000465514509826766,
0.0000744430049271689, \ 0.0000744451636606797,
0.0000744451636607048
```

This approximation is so much better than the brute-force method with the prime numbers that even for $n = 2$ it is already very close. And it is almost 150 times faster.

Possible generalisations
For this case we worked with the number theoretic character χ_4 and the corresponding L-series. It is possible to generalise the solution to other number theoretic characters. For example, for the functions ψ_5 and ψ_1, for which we would use the character χ_6, the following holds:

$$\psi_1(s) \ = \prod_p \frac{1}{\left(1-\frac{1}{p^s}\right)}$$

where p runs over all primes that are 1 modulo 6.

$$\psi_5(s) \ = \prod_q \frac{1}{\left(1-\frac{1}{q^s}\right)}$$

where q runs over all primes that are 5 modulo 6.
We can therefore approximate the ratio $\psi_5(1)/\psi_1(1)$ as follows:

- Choose a positive integer n.
- Approximate $\frac{\psi_5(2^n)}{\psi_1(2^n)}$ by supposing that $\frac{\psi_5(2^n)}{\psi_1(2^n)} = 1$
- Use the following recursive formula a number of times to obtain $\frac{\psi_5(1)}{\psi_1(1)}$:

$$\frac{\psi_5\left(2^{n-1}\right)}{\psi_1\left(2^{n-1}\right)} = \sqrt{\frac{\left(1-\frac{1}{2^{2^n}}\right)\left(1-\frac{1}{3^{2^n}}\right)\zeta\left(2^n\right)}{L_6\left(2^{n-1},\chi_6\right)^2}\frac{\psi_5\left(2^n\right)}{\psi_1\left(2^n\right)}}$$

The extra term with 3 is due to the fact that the character χ_6 excludes 2 and 3, which are not coprime with 6, while χ_4 only excludes 2.

Results of Session 2005/3

	Name	A	B	C	*Total*
1.	A.P.M. Kupers en J.W.T. Konter	10	6	8	94
2.	DESDA	6	-	-	18

Final Table after Session 2005/3
We give the top 3, the complete table can be found on the UWC website.

	Name	*Points*
1.	Hendrik Hubrechts	90
2.	Syb Botma	42
3.	DESDA	38