Birgit van Dalen
Mathematisch Instituut
Universiteit Leiden
Postbus 9512
2300 RA Leiden
dalen@math.leidenuniv.nl

Onderwijs

Card games with ideals

Een kaartspel als illustratie voor een identiteit over tweezijdige idealen? Het bestaat, en het laat zien hoe met inventiviteit en fantasie een abstracte variant van de Chinese reststelling tot leven komt. Birgit van Dalen, student wiskunde aan de Universiteit Leiden, legt uit.

Lenstra's Wondrous Card Game is a mathematical card game for two players. It was invented by Hendrik Lenstra in order to solve a problem in ring theory. To play the game you need a mathematical stack of cards, consisting of infinitely many cards that are coloured at both sides. There is a finite number $n \geq 2$ of colours, and the two sides of each card have different colours. As many cards as needed are available of each of the different types of cards, corresponding with the different pairs of colours. In addition to that, there is exactly one joker, which has only one side and can assume any of the n colours.

All else you need for this game is a pencil and a piece of paper. Before the start of the game, the two players agree on a certain number of colour sequences, and they write these sequences down. Each sequence should contain every colour exactly once, and the order of the colours in each sequence is determined by the players. These sequences, as well as their number, can change with every game that is played and can be used to vary the difficulty of the game. The first player's objective is to create one of these sequences with the cards, so the more sequences, the easier the game is for the first player.

When the sequences have been written down, the game starts. The first player, whom we will call player A, takes any finite number of cards and lays them in a long row. When he is finished, the second player, player B, for every card decides on a side to turn up. Only the colour on the side turned up will matter for the creation of the sequences.

After player B has finished, the winner is determined. If the row of cards, when read from left to right, contains one of the colour sequences agreed on in advance, player A wins the game. In matching these sequences, cards in the row may be skipped, as long as the relative order of the colours is preserved. The joker may be part of a sequence and will in that case assume any colour player A assigns to it. If there is no such sequence available in the row of cards, player B wins the game.

Before giving an example, we shall introduce some notation. The collection of colours will henceforth be called V. We had already agreed that V contains n elements. The collection of colour sequences will be called T. Rather than using the full names of the colours, we will use the numbers $1, 2, \ldots, n$ to indicate the colours.

Example 1. $n = 4$, $V = \{1, 2, 3, 4\}$, $T = \{1234, 4231, 1342\}$. Player A lays down the following row of cards. Each card is represented by a column containing the two colours on the card. The joker is represented by J.

$$\begin{matrix} 1 & 3 & 2 & 4 & 1 \\ 2 & 2 & 4 & 3 & 2 \end{matrix} \quad J \quad \begin{matrix} 3 & 1 \\ 2 & 2 \end{matrix}.$$

Player B may now choose the sides of the cards that will be turned up. If he chooses, for example, the row

$$1 \quad 3 \quad 2 \quad 4 \quad 1 \quad J \quad 3 \quad 1,$$

player A wins, since this row contains the sequence 4231 (with the joker as 2). If, on the other hand, player B chooses

$$2 \quad 3 \quad 2 \quad 3 \quad 2 \quad J \quad 3 \quad 2,$$

he wins: the row contains neither colour 4 nor colour 1, and as the joker can only assume one of those colours and each sequence in T contains all four colours, this row can never contain a sequence.

A condition on T

We would now like to know which of the players can win this game. This depends heavily on the set T. Suppose, for example, that T is empty. Then it follows directly from the rules that player B wins the game. On the other hand we can also easily construct a T such that player A wins the game.

Example 2. $T = \{123 \ldots n, n \ldots 321\}$. For $n = 2$ we have $T = \{12, 21\}$ and we see immediately that player A wins with the row of cards

$$J\begin{array}{c}1\\2\end{array}.$$

We now use induction to n to show that player A can always win in this case. Suppose Z_{n-1} is a winning row of cards for $T = \{123 \ldots (n-1), (n-1) \ldots 321\}$. Now take $T = \{123 \ldots n, n \ldots 321\}$ and observe the following row of cards:

$$\begin{array}{ccccc}n & n & n & \cdots & n\\ 1 & 2 & 3 & \cdots & n-1\end{array}\; Z_{n-1}\; \begin{array}{ccccc}n & \cdots & n & n & n\\ n-1 & \cdots & 3 & 2 & 1\end{array}.$$

First note that Z_{n-1} contains the joker: if it did not, player B could pick one colour and turn that to the bottom on each card it appears on, so this colour would not be visible in the row and therefore none of the sequences would appear.

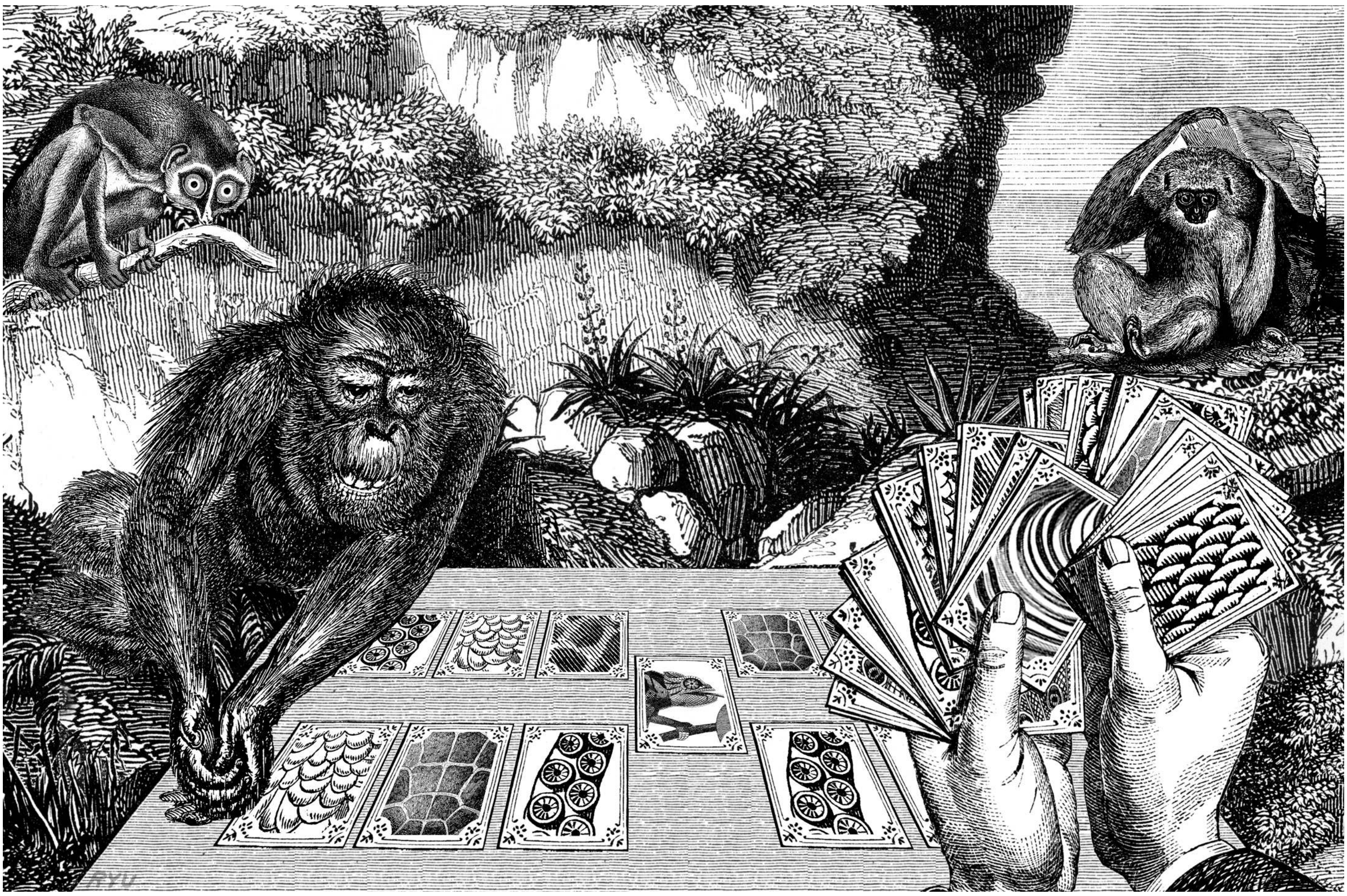
Now look at the left side of the row of cards. If player B chooses the colours $1\ 2\ 3\ \ldots\ n-1$, the joker can assume colour n and player A will win with the sequence $123 \ldots n$. So player B is forced to choose colour n at least once on the left side of Z_{n-1}. In the same way, player B is forced to choose n at least once on the right side of Z_{n-1}. Now we know that Z_{n-1} is a winning row for $T = \{123 \ldots (n-1), (n-1) \ldots 321\}$, meaning that at least one of those sequences appears in Z_{n-1}, regardless of what player B does. Together with the n left and right, we now see that either $123 \ldots n$ or $n \ldots 321$ appears in the full row of cards. So player A always wins.

Let's have a closer look at the argument used here to show that player A always needs the joker. If player B chooses one colour, say 1, and turns all cards with this colour so that the side with 1 is not visible, then player A is forced to make the joker assume colour 1. This is a severe restriction, since to construct a sequence τ every colour to the left of 1 in τ will now have to appear to the left of the joker in the row of cards, and every colour to the right of 1 in τ will now have to appear to the right of the joker in the row.

We introduce a notation for this. Suppose we view τ as a function that assigns a colour to every place number of the sequence, such that $\tau(1)$ is the first colour in the sequence, $\tau(2)$ the second, and so on, with $\tau(n)$ the final colour. The colours left of 1 in the sequence τ (or rather, the sequence $\tau(1)\ \tau(2)\ \ldots\ \tau(n)$) we can now write as $\tau(i)$ for each $i < \tau^{-1}(1)$. Similarly, the colours right of 1 in τ are $\tau(i)$ for each $i > \tau^{-1}(1)$.

If player B could force A to use the joker for two different colours, he would win, since that is impossible. Unfortunately for player B, things are not that easy. As long as each type of the cards occurs at least once in the row, player B cannot let two different colours vanish. However, he can split the row in two halves and let one colour vanish in each half. The most natural way to do this is obviously to split the row at the joker.

What happens if player B turns colour 1 to the bottom of all cards to the left of the joker and does the same to colour 2 to the right of the joker? In order to construct a sequence, player A will have



illustratie: Ryu Tajiri

to pick either the joker or a card to the right of the joker for colour 1, and either the joker or a card to the left of the joker for colour 2. This means that whatever player A does, colour 1 will always be to the right of colour 2 in the sequence he constructs. In other words, player A can only win if there is a sequence τ that satisfies $\tau^{-1}(2) < \tau^{-1}(1)$.

Player B has a huge advantage here. He can analyse the collection of sequences T and pick two colours he likes to execute the above trick with. So for player A to stand a chance, T needs to satisfy the following condition:

Condition $(*)$. *For all colours $v, w \in V$ with $v \neq w$ there exists a $\tau \in T$ such that $\tau^{-1}(v) < \tau^{-1}(w)$.*

How player A can win

In example 2 we have seen a set T for which player A can win. It is immediately obvious that this T satisfies condition $(*)$. In fact, this T is the smallest collection of sequences that satisfies the condition. This means of course that this T has special properties, but intuitively we feel that the bigger T, the easier player A can win. Therefore, we hope to find a way for player A to win every game with a T that satisfies condition $(*)$.

Let us look carefully at the way player A wins in example 2. We notice that an important special property of T in example 2 is that colour n occurs both at the start of a sequence and at the end of a sequence. This allows player A to force player B to choose n both to the left and to the right of the joker. We can generalise this a little.

Suppose $a \in V$ is a colour for which there exists a τ such that $\tau(1) = a$. Then player A can make sure that either he wins or player B chooses colour a right of the joker at a point in the row of cards determined by player A. In other words, player A can make sure colour a appears wherever he wants, as long as it is to the right of the joker.

Player A can do this by inserting the following set of cards in the place he likes to the right of the joker:

$$\begin{matrix} a & a & \cdots & a \\ \tau(2) & \tau(3) & \cdots & \tau(n) \end{matrix}$$

If player B does not choose colour a anywhere in these cards, he has to choose $\tau(2), \tau(3), \ldots, \tau(n)$. Since these cards are lying somewhere to the right of the joker, the sequence τ is formed, with the joker assuming colour $a = \tau(1)$. So in order to prevent player A from winning, player B has to choose colour a at least once.

As we see, player A can win if he is able to execute the above trick with every colour. After all, in that case he can simply force player B to choose each of the colours to the right of the joker in the order they appear in one of the sequences. Unfortunately, there are numerous sets T for which this does not work, like the one from example 2.

However, we can generalise a bit more. Suppose colour a does not appear on the first place of a sequence τ, but on the second: $\tau(2) = a$. Then we can put the following cards to the right of the joker:

$$\begin{matrix} a & a & \cdots & a \\ \tau(3) & \tau(4) & \cdots & \tau(n) \end{matrix}$$

Now we are nearly there. If player B does not choose a, the sequence τ will nearly appear; only $\tau(1)$ and $\tau(2) = a$ are not there yet. Colour a, of course, can be made by the joker. All we now need to complete the sequence is colour $\tau(1)$, say b, to the left of the joker. If we can force player B to choose b to the left of the joker, he will also have to choose colour a to the right.

This is where condition $(*)$ comes in. We know that there is at least one sequence in T in which a appears to the left of b and at least one in which a appears to the right of b. Clearly, τ satisfies the latter. We also pick a sequence σ that satisfies the former. Let's call $\sigma^{-1}(a) = m$. Now put

$$\begin{matrix} a & a & \cdots & a \\ \sigma(1) & \sigma(2) & \cdots & \sigma(m-1) \end{matrix}$$

to the left of the joker and

$$\begin{matrix} a & a & \cdots & a \\ \sigma(m+1) & \sigma(m+2) & \cdots & \sigma(n) \end{matrix}$$

to the right of the joker next to the cards we already had there. Note that all of these cards exist: as $\sigma(m) = a$, none of the colours $\sigma(i)$ with $i > m$ is equal to a, and as $\sigma^{-1}(b) > \sigma^{-1}(a) = m$, none of the colours $\sigma(i)$ with $i < m$ is equal to b.

Now if player B chooses neither a to the right of the joker nor b to the left of the joker, the sequence σ will appear. If player B chooses b to the left of the joker but not a to the right of the joker, the sequence τ will appear. So the only thing left for player B to do is to choose a to the right of the joker.

We now see that if all n colours occur in either the first or the second position in a sequence of T, player A can force player B to choose them all to the right of the joker in the order of one of the sequences of T. Then player A can win.

This is an encouraging result. Not only have we been able to generalise a useful principle that we may be able to generalise even further, we have also explicitly used the condition we imposed upon the set T. We may now hope that this condition is not only necessary, but also sufficient for player A to win the game.

So what to do if a colour a occurs not on the first or second position in a sequence τ, but even further to the right, say on the kth position? We then have $k-1$ colours left of a in τ. Let's call them $b_1, b_2, \ldots, b_{k-1}$. We now want to use condition $(*)$ and find a sequence σ that has all of these colours to the right of a. Unfortunately, this is not always possible. All condition $(*)$ says is that for each of these colours b_i there exists a sequence $\sigma_i \in T$ such that $\sigma_i^{-1}(b_i) > \sigma_i^{-1}(a)$, but these sequences are not necessarily all the same.

We are in luck, however. With a few more cards, we can still make it work. Let's call $\sigma_i^{-1}(a) = m_i$. We now do for each of the b_i exactly the same thing as we did above for b. We use the following sets of cards:

$$H_i = \begin{matrix} \sigma_i(1) & \sigma_i(2) & \ldots & \sigma_i(m_i-1) \\ b_i & b_i & \ldots & b_i \end{matrix} \quad \text{for } k < i \leq n,$$

$$K_i = \begin{matrix} \sigma_i(m_i+1) & \sigma_i(m_i+2) & \ldots & \sigma_i(n) \\ a & a & \ldots & a \end{matrix} \quad \text{for } k < i \leq n.$$

Note that we have chosen σ_i such that all of these cards exist.

If we lay down H_i to the left of the joker and K_i to the right of the joker, player B will be forced to choose either b_i to the left of the joker or a to the right of the joker. We do this for each i, $i < k$, where $k = \tau^{-1}(a)$. If player B refused to choose a to the right of the joker, the first half of τ would appear to the left of the joker. Now we just need to add the other half of τ at the right side:

$$L = \begin{array}{cccc} \tau(k+1) & \tau(k+2) & \dots & \tau(n) \\ a & a & \dots & a \end{array}.$$

The complete row of cards now looks like the following. J represents the joker.

$$H_1 \quad H_2 \quad \dots \quad H_{k-1} \quad J \quad K_1 \quad K_2 \quad \dots \quad K_{k-1} \quad L.$$

If player B does not choose a to the right of the joker, player A will win, either because one of the σ_i appears, or because player B chooses $b_1, b_2, \dots, b_{k-1}$ to the left and $\tau(k+1), \tau(k+2), \dots, \tau(n)$ to the right, forming the sequence τ.

This can be done for each colour as long as T satisfies $(*)$, so by putting the blocks of cards on the right side of the joker in a certain order player A can make sure one of the sequences of T appears in the row of cards.

It may seem as though this method requires huge numbers of cards. It is not nearly as bad as it seems, however. Each pair H_i and K_i consists of at most $n-1$ cards, and there are at most $n-1$ of these pairs. The final set of cards L also contains at most $n-1$ cards, so for one colour you need fewer than n^2 cards (not counting the joker). Since there are only n colours, the total number of cards will never exceed n^3. In practice you need even fewer cards, as the following example illustrates.

Example 3. Take $n = 5$ and $T = \{12345, 45132, 21354, 42315\}$. This T satisfies condition $(*)$, so player A can win. To do so, we first take $a = 3$ and $\tau = 12345$. To the left of 3 in this sequence are 1 and 2. Colour 1 appears to the right of colour 3 in sequence $\sigma_1 = 42315$; colour 2 appears to the right of colour 3 in sequence $\sigma_2 = 45132$. Player A can now force player B to choose colour 3 to the right of the joker by laying down the following row of cards:

$$\begin{array}{cc} 4 & 2 \\ 1 & 1 \end{array} \quad \begin{array}{ccc} 4 & 5 & 1 \\ 2 & 2 & 2 \end{array} \quad J \quad \begin{array}{cc} 1 & 5 \\ 3 & 3 \end{array} \quad \begin{array}{c} 2 \\ 3 \end{array} \quad \begin{array}{cc} 4 & 5 \\ 3 & 3 \end{array}$$

It is much easier to let 1 and 2 appear to the right of the joker, as both of these colours occur at the beginning of a sequence:

$$J \; \begin{array}{cccc} 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 \end{array}, \qquad J \; \begin{array}{cccc} 1 & 3 & 5 & 4 \\ 2 & 2 & 2 & 2 \end{array}.$$

On the other side of the joker we can do the same with 4, as that colour occurs at the end of a sequence:

$$\begin{array}{cccc} 2 & 1 & 3 & 5 \\ 4 & 4 & 4 & 4 \end{array} \; J.$$

Now we are already done. We can construct the sequence 45132 by using all of the above and making the joker assume colour 5. So player A can win this game with the following row of cards:

$$\begin{array}{ccccccccc} 4 & 2 & 4 & 5 & 1 & 2 & 1 & 3 & 5 \\ 1 & 1 & 2 & 2 & 2 & 4 & 4 & 4 & 4 \end{array} \; J \; \begin{array}{ccc} 2 & 3 & 4 \\ 1 & 1 & 1 \end{array} -$$

$$\begin{array}{cccccccccc} 5 & 1 & 5 & 2 & 4 & 5 & 1 & 3 & 5 & 4 \\ 1 & 3 & 3 & 3 & 3 & 3 & 2 & 2 & 2 & 2 \end{array}.$$

The connection with algebra

Now that we know exactly what condition T must satisfy to allow player A to win, we can easily solve a certain algebraic problem. Let R be an arbitrary ring and let I, J be two-sided ideals of R that are coprime. This means that we can find elements $x \in I$ and $y \in J$ such that $x + y = 1$. We now wish to find a relation between the intersection $I \cap J$ and the products IJ and JI. This is quite easy. Suppose $i \in I$ and $j \in J$ are arbitrary elements. One of the main properties of ideals says that $ir \in I$ for all $r \in R$; in particular, $ij \in I$. Similarly, $rj \in J$ for all $r \in R$, hence $ij \in J$. So $ij \in I \cap J$ for all $i \in I$ and $j \in J$; in other words,

$$IJ \subset I \cap J.$$

Similarly,

$$JI \subset J \cap I = I \cap J,$$

and from this it follows that

$$IJ + JI \subset I \cap J. \tag{1}$$

Now let z be an arbitrary element in $I \cap J$. Now

$$z = 1 \cdot z = (x+y)z = xz + yz \in IJ + JI$$

since $xz \in IJ$ and $yz \in JI$. So

$$I \cap J \subset IJ + JI$$

and hence

$$I \cap J = IJ + JI.$$

Now we can ask ourselves: is there a shorter way to write $I \cap J$ in terms of the products? Perhaps $I \cap J = IJ$? We want this to hold for all rings R and all ideals I and J, however, so one counterexample is sufficient to disprove this claim. Such a counterexample indeed exists.

Example 4. Let $n = 2$. As ring R we take a subring of the ring of 2×2-matrices with entries in $\mathbf{Z}$:

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbf{Z} \right\}.$$

It is easy to verify that this is indeed a subring. Now we define two two-sided ideals of R:

$$I = \left\{ \begin{pmatrix} 0 & b \\ 0 & c \end{pmatrix} : b, c \in \mathbf{Z} \right\}.$$

$$J = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in \mathbf{Z} \right\}.$$

These ideals are coprime, since $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in I$ and $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in J$ and therefore $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in I + J$, which means that $I + J = R$.

Now for $\begin{pmatrix} 0 & b \\ 0 & c \end{pmatrix} \in I$ and $\begin{pmatrix} d & e \\ 0 & 0 \end{pmatrix} \in J$ it holds that

$$\begin{pmatrix} 0 & b \\ 0 & c \end{pmatrix}\begin{pmatrix} d & e \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

so $IJ = \{0\}$. However,

$$I \cap J = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} : b \in \mathbf{Z} \right\} \neq \{0\}.$$

We conclude that $I \cap J \neq IJ$.
We have now satisfactorily solved this problem with two ideals. Things become more interesting, however, if we look at n ideals. So let R once again be an arbitrary ring and let $I_1, I_2, \ldots, I_n$ be pairwise coprime two-sided ideals. How can we express $I_1 \cap I_2 \cap \ldots \cap I_n$ in terms of the products of the ideals?
This time we have many more possibilities. There are $n!$ ways to order the n ideals in a product, and we may have to add two or more of them. In other words, we are looking for a subset $T \subset S_n$ (with S_n the permutation group) such that

$$I_1 \cap I_2 \cap \ldots \cap I_n = \sum_{\tau \in T} I_{\tau(1)} \cdots I_{\tau(n)} \tag{2}$$

for all rings R and all pairwise coprime two-sided ideals $I_1, I_2, \ldots, I_n$ of R.
Using (1), we can show by induction that

$$I_1 \cap I_2 \cap \ldots \cap I_n \supset \sum_{\tau \in T} I_{\tau(1)} \cdots I_{\tau(n)} \tag{3}$$

for all subsets $T \subset S_n$.
We named the subset of permutations T on purpose. If we view the permutations as orders rather than as functions, we will see that T is a collection of sequences of ideals: the permutation τ corresponds with the sequence $I_{\tau(1)} \cdots I_{\tau(n)}$. So this T looks very similar to the T we used in the card game, and we will in fact prove that the condition (2) on T is equivalent to $(*)$.
For the first step, we need example 4. Suppose there exist $v, w \in \{1, 2, \ldots n\}$ such that for each $\tau \in T$ it holds that $\tau^{-1}(v) < \tau^{-1}(w)$. Then (2) is not true for all rings R and all pairwise coprime two-sided ideals $I_1, I_2, \ldots, I_n$ of R.
We can prove this claim as follows. Take R as in the above example. The assumption on T implies that in each product on the right-hand side of (2), the ideal I_v appears left of the ideal I_w. Now take $I_v = I$ and $I_w = J$ as in the above example, and $I_i = R$ for all other i. The right-hand side of (2) is now reduced to IJ while the left-hand side is reduced to $I \cap J$. We already knew that $I \cap J \neq IJ$, and this proves our claim.
So now we know that condition $(*)$ is a necessary condition on T for (2) to hold. All that is left to prove is that it is also a sufficient condition. We will not prove this directly, but instead use the card game. If we can show that player A being able to win the card game means that (2) holds, we are done, since condition $(*)$ is a sufficient condition for player A being able to win the card game. Rather than proving this formally, we use an example to illustrate the way it should be proved.

Example 5. Take $n = 3$ and $T = \{123, 312, 231\}$. By using the techniques we have developed, we can easily construct a winning row of cards:

$$\begin{matrix} 2 \\ 3 \end{matrix} \quad \begin{matrix} 2 \\ 1 \end{matrix} \quad J \quad \begin{matrix} 1 \\ 2 \end{matrix} \quad \begin{matrix} 1 \\ 3 \end{matrix}.$$

The fact that this is a winning row of cards means that regardless of the choices player B makes, there will always appear a sequence of T.
Now we take an arbitrary ring R and three pairwise coprime two-sided ideals I_1, I_2, I_3. Since each pair of them is coprime, we can find elements $x_1, y_1 \in I_1$, $x_2, y_2 \in I_2$, $x_3, y_3 \in I_3$ such that

$$y_1 + x_2 = 1, y_2 + x_3 = 1, y_3 + x_1 = 1.$$

Now we go back to the row of cards and replace each card by 1: instead of $\begin{matrix} 2 \\ 3 \end{matrix}$ we write $(y_2 + x_3)$, instead of $\begin{matrix} 2 \\ 1 \end{matrix}$ we write $(x_2 + y_1)$, and so on. We replace the joker by an arbitrary element $r \in I_1 \cap I_2 \cap I_3$. So the row of cards now looks like a product:

$$(y_2 + x_3)(x_2 + y_1)r(y_1 + x_2)(x_1 + y_3).$$

All sums between parentheses are equal to 1, so the entire expression is equal to r. On the other hand, multiplying out the parentheses is the same as picking a side of each card except the joker, so the resulting products all contain a sequence of T. Similar to the way we can let the joker assume colour i in the card game if we wish, we can view r as an element of I_i for any i.
Now we use the fact that $xI \subset I$ and $Ix \subset I$ for all elements $x \in R$ and two-sided ideals I of R. So if in a product the sequence $\tau \in T$ appears, we can disregard any elements in the product not contributing to τ and conclude that the product is an element of $I_{\tau(1)} I_{\tau(2)} I_{\tau(3)}$. Since each product contains a $\tau \in T$, the entire expression is an element of $\sum_{\tau \in T} I_{\tau(1)} I_{\tau(2)} I_{\tau(3)}$. So

$$I_1 \cap I_2 \cap I_3 \subset \sum_{\tau \in T} I_{\tau(1)} I_{\tau(2)} I_{\tau(3)}.$$

On the other hand, it follows from (3) that

$$I_1 \cap I_2 \cap I_3 \supset \sum_{\tau \in T} I_{\tau(1)} I_{\tau(2)} I_{\tau(3)}.$$

This proves

$$I_1 \cap I_2 \cap I_3 = \sum_{\tau \in T} I_{\tau(1)} I_{\tau(2)} I_{\tau(3)}.$$

It is clear that the same argument can be used for all n and T for which there exists a winning row of cards. We have obtained the following result:
Let $n \geq 2$ and $T \subset S_n$, which we can view as a set of sequences. Then the following claims are equivalent:

(i) for all rings R and all pairwise coprime two-sided ideals $I_1, \ldots, I_n$ of R the identity: $\bigcap_{i=1}^n I_i = \sum_{\tau \in T} I_{\tau(1)} \cdots I_{\tau(n)}$ holds;
(ii) player A can win the cardgame defined by T;
(iii) for all $v, w \in \{1, 2, \ldots n\}$ with $v \neq w$ there exists $\tau \in T$ such that $\tau^{-1}(v) < \tau^{-1}(w)$.