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Profinite Fibonacci numbers

Goede recreatieve wiskunde doet je meteen naar pen en papier grijpen. Het onderwerp moet eenvoudig en aanstekelijk zijn, het liefst nog met allerlei raadselachtige eigenschappen binnen handbereik. Hendrik Lenstra breekt een lans voor pro-eindige getallen. Rekent u mee?

Profinite integers do not enjoy widespread popularity among mathematicians. They form an important technical tool in several parts of algebraic number theory and arithmetic geometry, but their recreational virtues have never been recognized. The purpose of the present paper is to acquaint the casual mathematical reader in an informal way with profinite integers and some of their remarkable properties. The less casual reader is warned that the approach is experimental and heuristic, and that the exact meaning of many assertions may not always be instantly clear. Providing not only precise formulations, but also valid proofs, is a challenge that an expert in p-adic numbers and their analysis can easily face, but that hardly does justice to the entertainment value of the subject.

To define profinite integers, we recall that any positive integer n has a unique representation of the form

$$n = c_k \cdot k! + c_{k-1} \cdot (k-1)! + \ldots + c_2 \cdot 2! + c_1 \cdot 1!,$$

where the 'digits' c_i are integers satisfying $c_k \neq 0$ and $0 \leq c_i \leq i$, for $1 \leq i \leq k$. In the *factorial number system*, the number n is then written as

$$n = (c_k c_{k-1} \ldots c_2 c_1)_! . \tag{1}$$

The exclamation mark distinguishes the factorial representation from the decimal representation. For example, we have $5 = (21)_!$ and $25 = (1001)_!$.

If we allow the sequence of digits to extend indefinitely to the left, then we obtain a *profinite integer*:

$$(\ldots c_5 c_4 c_3 c_2 c_1)_!,$$

where we still require $0 \leq c_i \leq i$ for each i. Usually, only a few of the digits are specified, depending on the accuracy that is required. In this paper, most profinite numbers are given to an accuracy of 24 digits. For example, we shall encounter the following profinite integer:

$$l = (\ldots {}^1\!6\,0\,4\,{}^1\!6\,{}^1\!3\,{}^1\!8\,{}^1\!0\,4768\,{}^1\!0\,{}^1\!0\,49000120100)_! . \tag{2}$$

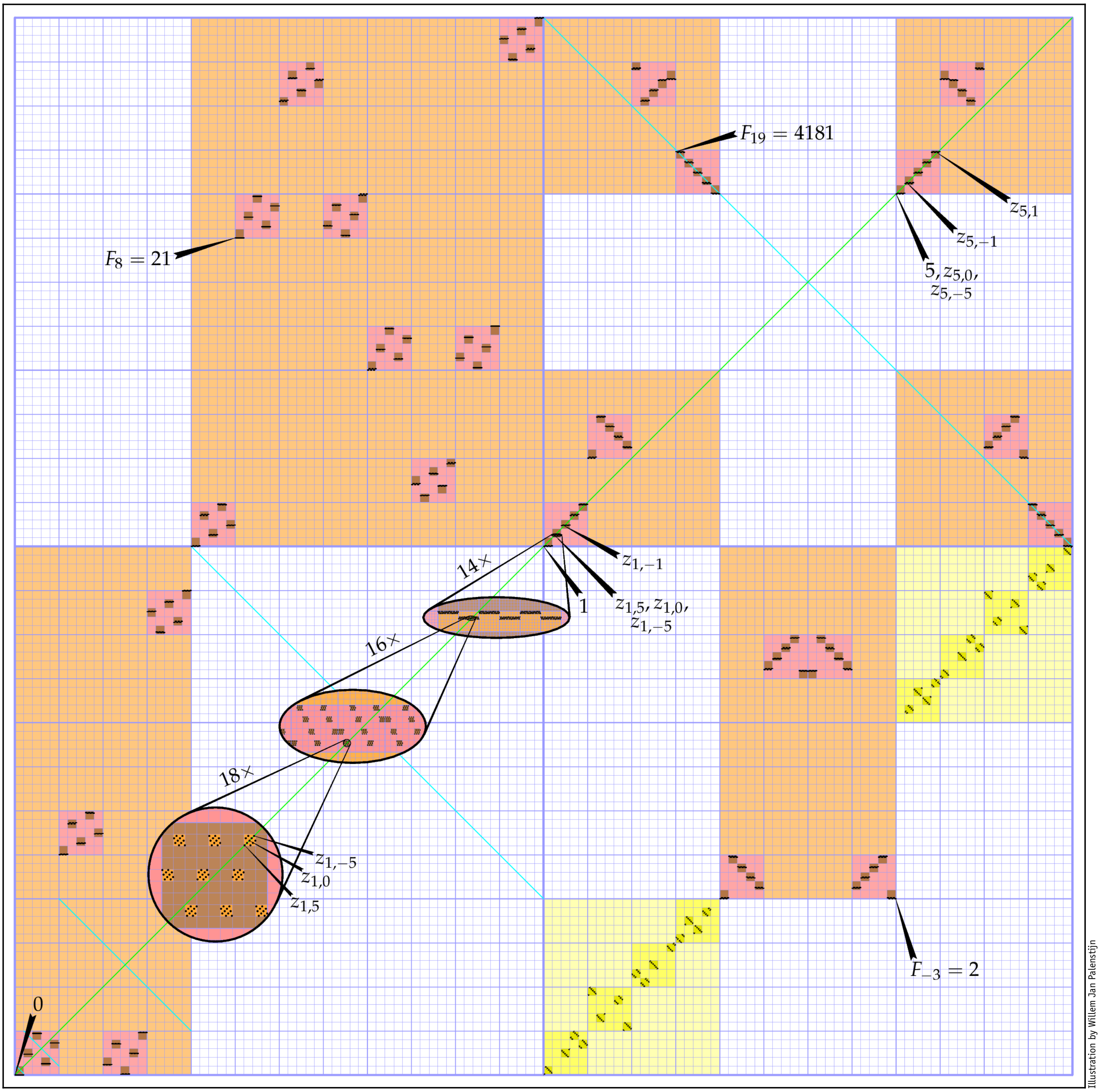
In this number, the 19th digit has the value 18, but this is written ${}^1\!8$ in order to express that it is a single digit. Note that by the 19th digit we mean the 19th digit *from the right*. Likewise, when we speak about the 'first' digits or the 'initial' digits of a profinite number, we always start counting from the right.

One can view each positive integer n as in (1) as a profinite integer, by taking $c_i = 0$ for $i > k$. Also 0 is a profinite integer, with *all* digits equal to 0. The *negative* integers can be viewed as profinite integers as well, for example

$$-1 = (\ldots {}^2\!4\,{}^2\!3\,{}^2\!2\,{}^2\!1\,{}^2\!0\,{}^1\!9\,{}^1\!8\,{}^1\!7\,{}^1\!6\,{}^1\!5\,{}^1\!4\,{}^1\!3\,{}^1\!2\,{}^1\!1\,{}^1\!0\,987654321)_!,$$

with $c_i = i$ for all i. In general, negative integers are characterized by the property that $c_i = i$ for all but finitely many i.

The ordinary arithmetic operations can be performed on profinite integers. To add two profinite integers, one adds them digit-



This picture shows the graph of the Fibonacci function $\hat{\mathbf{Z}} \to \hat{\mathbf{Z}}$. Each element $(\ldots c_3c_2c_1)_! = \sum_{i\geq 1} c_i i!$ of $\hat{\mathbf{Z}}$ is represented by the number $\sum_{i\geq 1} c_i/(i+1)!$ in the unit interval, and the graph $\{(s, F_s) : s \in \hat{\mathbf{Z}}\}$ is correspondingly represented as a subset of the unit square. Successive approximations to the graph are shown in orange, red, and brown. Intersecting the diagonal, shown in green, with the graph, one finds the eleven fixed points 0, 1, 5, $z_{1,-5}$, ..., $z_{5,1}$ of the function. There are two clusters of three fixed points each that are indistinguishable in the precision used. One of these clusters is resolved by a sequence of three blow-ups, with a total magnification factor of $14 \times 16 \times 18 = 4032$. The graph of the function $s \mapsto -s$, shown in blue, enables the viewer to check the formula $F_{-s} = (-1)^{s-1}F_s$. The yellow squares contain the curve $\{(s,t) \in \hat{\mathbf{Z}} \times \hat{\mathbf{Z}} : s \cdot (t+1) = 1\}$, which projects to the group of units of $\hat{\mathbf{Z}}$ on the horizontal axis.

wise, proceeding from the right; when the sum of the ith digits is found to exceed i, one subtracts $i+1$ from it and adds a carry of 1 to the sum of the $i+1$st digits. The reader may check that in this way one finds that the sum of 1 and -1 equals 0. Subtraction is performed in a similar manner. Multiplication can be done by means of a more elaborate scheme, but it is often more practical to compute products by means of the following rule: for each k, the first k digits of the product of two profinite numbers s and t depend only on the first k digits of s and of t. (This rule is also valid for addition and subtraction.) Using this rule, one reduces the problem of computing products to the case of ordinary positive integers. These operations make the set of all profinite integers into a commutative ring with unit element 1. This ring is denoted $\hat{\mathbf{Z}}$, the *ring of profinite integers*.

Fibonacci numbers

Fibonacci numbers illustrate several features of profinite integers. The nth Fibonacci number F_n is, for $n \geq 0$, inductively defined by $F_0 = 0$, $F_1 = 1$, and

$$F_n = F_{n-1} + F_{n-2} \tag{3}$$

for $n > 1$. It is well known that one can extend the definition to negative n by putting $F_n = (-1)^{n-1} F_{-n}$, and that many familiar identities, such as (3) and

$$F_n F_{m+1} - F_{n+1} F_m = (-1)^m \cdot F_{n-m}, \tag{4}$$

then hold for *all* integers n and m. There is, however, no reason to stop here.

For each profinite integer s, one can in a natural way define the sth Fibonacci number F_s, which is itself a profinite integer. Namely, given s, one can choose a sequence of positive integers n_1, n_2, $n_3, \ldots$ that share more and more initial digits with s, so that it may be said that n_i *converges* to s for $i \to \infty$. Then the numbers F_{n_1}, F_{n_2}, $F_{n_3}, \ldots$ share more and more initial digits as well, and we define F_s to be their 'limit' as $i \to \infty$. This does not depend on the choice of the sequence of numbers n_i.

For example, we can write $s = -1$ as the limit of the numbers $n_1 = (21)_! = 5$, $n_2 = (321)_! = 23$, $n_3 = (4321)_! = 119$, $n_4 = (54321)_! = 719, \ldots$, so that F_{-1} is the limit of

$$\begin{aligned} F_5 &= 5 = (21)_!, \\ F_{23} &= 28657 = (5444001)_!, \\ F_{119} &= 3311648143516982017180081 \\ &= (5826{}^{1}_{4}{}^{1}_{1}8{}^{1}_{0}{}^{1}_{5}{}^{1}_{3}23418173200001)_!, \\ F_{719} &= (\ldots 3{}^{1}_{6}{}^{1}_{6}98{}^{1}_{6}{}^{1}_{2}51{}^{1}_{1}{}^{1}_{4}3{}^{1}_{1}49806000001)_!, \\ &\ldots, \end{aligned}$$

which is consistent with the true value $F_{-1} = 1 = (\ldots 000001)_!$.

For each $k \geq 3$ the first k digits of F_s are determined by the first k digits of s. This rule makes it possible to compute profinite Fibonacci numbers, as we shall see below.

Many identities such as (3) and (4) are also valid for profinite Fibonacci numbers. In order to give a meaning to the sign that appears in (4), we call a profinite integer s *even* or *odd* according as its first digit c_1 is 0 or 1, and we define $(-1)^s = 1$ or -1 accordingly. More generally, one defines a profinite integer s to be *divisible* by a positive integer b if the factorial number formed by the first $b-1$ digits of s is divisible by b. For many b, it suffices to look at far fewer than $b-1$ digits. For example, if k is a non-negative integer, then a profinite integer is divisible by $k!$ if and only if its $k-1$ initial digits are zero. Two profinite numbers s_1 and s_2 are called *congruent* modulo a positive integer b if their difference is divisible by b, notation: $s_1 \equiv s_2 \bmod b$.

The following method may be used to compute profinite Fibonacci numbers. Let s be a profinite number, and suppose that one wishes to compute the sth Fibonacci number F_s to an accuracy of k digits, for some $k \geq 3$. Then one first truncates s to k digits, which gives a non-negative integer n that is usually very large. By the rule mentioned above, F_s and F_n share at least k initial digits, so it suffices to calculate F_n to a precision of k digits. To this end, let ϑ be a symbol that satisfies the rule $\vartheta^2 = \vartheta + 1$. Then for all n one has $\vartheta^n = F_n \vartheta + F_{n-1}$. The left hand side can be quickly calculated by induction, even for very large n, if one uses the identities $\vartheta^{2m} = (\vartheta^m)^2$ and $\vartheta^{2m+1} = \vartheta^{2m} \cdot \vartheta$. All intermediate results are expressed in the form $a\vartheta + b$, where a and b are integers that are only computed to a precision of k digits in the factorial number system. Then in the end one knows F_n to a precision of k digits as well, as required.

The *Lucas numbers* L_n, which are defined by $L_0 = 2$, $L_1 = 1$, $L_n = L_{n-1} + L_{n-2}$ ($n > 1$), can be generalized to profinite numbers in a completely similar manner. They are expressed in Fibonacci numbers by $L_s = F_{s+1} + F_{s-1}$. One has also $F_s L_s = F_{2s}$ for all $s \in \hat{\mathbf{Z}}$; however, it is not necessarily meaningful to write $L_s = F_{2s}/F_s$, since division is not always well-defined in $\hat{\mathbf{Z}}$.

Power series expansions

A striking property of profinite Fibonacci numbers is that they have *power series* expansions. If $s_0 \in \hat{\mathbf{Z}}$, then the power series expansion for F_s around s_0 takes the shape

$$\begin{aligned} F_s &= F_{s_0} + l L_{s_0}(s - s_0) + 5l^2 F_{s_0} \frac{(s-s_0)^2}{2!} \\ &\quad + 5l^3 L_{s_0} \frac{(s-s_0)^3}{3!} + 5^2 l^4 F_{s_0} \frac{(s-s_0)^4}{4!} + \ldots \\ &= \sum_{i=0}^{\infty} \Big(5^i l^{2i} F_{s_0} \frac{(s-s_0)^{2i}}{(2i)!} + 5^i l^{2i+1} L_{s_0} \frac{(s-s_0)^{2i+1}}{(2i+1)!} \Big), \end{aligned} \tag{5}$$

where l is a certain profinite integer that is given by (2). The number l is divisible by all prime numbers except 5. From this it follows that $5^i l^{2i}$ and $5^i l^{2i+1}$ are divisible by $(2i)!$ and $(2i+1)!$, respectively, so that the coefficients in the power series expansions are profinite integers.

No prime number p is known for which l is divisible by p^2. In fact, if p is a prime number, then the number of factors p in l is the same as the number of factors p in $F_{p-1}F_{p+1}$, and no prime number is known for which $F_{p-1}F_{p+1}$ is divisible by p^2. One may, however, reasonably conjecture that there exist infinitely many such primes.

An informal derivation of (5) can be given as follows. Let again ϑ be such that $\vartheta^2 = \vartheta + 1$, and put $\vartheta' = 1 - \vartheta$. Then for all integers n one has $F_n = (\vartheta^n - \vartheta'^n)/(\vartheta - \vartheta')$ and $L_n = \vartheta^n + \vartheta'^n$. This suggests that one has $F_s = (\vartheta^s - \vartheta'^s)/(\vartheta - \vartheta')$ and $L_s = \vartheta^s + \vartheta'^s$ for all profinite integers s as well, and with a suitable interpretation of the powering operation this is indeed correct. Now consider the Taylor series for F_s around s_0:

$$F_s = \sum_{j=0}^{\infty} F_{s_0}^{(j)} \frac{(s-s_0)^j}{j!},$$

where $F_s^{(j)} = \frac{d^j F_s}{ds^j}$ denotes the jth derivative. To calculate these higher derivatives, one first notes that from $\vartheta\vartheta' = -1$ one obtains

$$2(\log \vartheta + \log \vartheta') = 2\log(-1) = \log 1 = 0,$$

and therefore $\log \vartheta = -\log \vartheta'$. This leads to

$$\begin{aligned} \frac{dF_s}{ds} &= \frac{d}{ds} \frac{\vartheta^s - \vartheta'^s}{\vartheta - \vartheta'} = \frac{\log \vartheta}{\vartheta - \vartheta'} \left(\vartheta^s + \vartheta'^s \right) = \frac{\log \vartheta}{\vartheta - \vartheta'} L_s, \\ \frac{dL_s}{ds} &= \log \vartheta \cdot \left(\vartheta^s - \vartheta'^s \right) = \log \vartheta \cdot (\vartheta - \vartheta') \cdot F_s. \end{aligned}$$

Combining this with $(\vartheta - \vartheta')^2 = 5$, one finds

$$F_s^{(2i)} = 5^i l^{2i} F_s, \qquad F_s^{(2i+1)} = 5^i l^{2i+1} L_s$$

for each $i \geq 0$, where

$$l = \frac{\log \vartheta}{\vartheta - \vartheta'}. \tag{6}$$

This leads immediately to (5).

If one makes this informal argument rigorous, using an appropriate theory of logarithms, then one discovers that the precise meaning of (5) is a little more subtle than one may have expected. Namely, one should interpret (5) to mean that, for each positive integer b, the following is true for every profinite integer s that shares sufficiently many initial digits with s_0: if k is any positive integer, then all but finitely many terms of the infinite sum are divisible by b^k, and the sum of the remaining terms is congruent to F_s modulo b^k. For example, if b divides $5! = 120$, then it suffices for s to share three initial digits with s_0, and if b divides $36!$ then six initial digits are enough.

One application of the power series development is the determination of l to any desired precision. Namely, put $s_0 = 0$, so that $F_{s_0} = 0$ and $L_{s_0} = 2$. Then the power series development reads

$$F_s = 2ls + \frac{2 \cdot 5 \cdot l^3 \cdot s^3}{3!} + \frac{2 \cdot 5^2 \cdot l^5 \cdot s^5}{5!} + \ldots . \tag{7}$$

Suppose that one wishes to determine the first 35 digits of l, or, equivalently, the residue class of l modulo $36!$. Modulo any power of $36!$, the expansion is valid for profinite numbers s of which the first six digits are zero. Choose

$$s = 2^{16} \cdot 3^8 \cdot 5^4 \cdot 7 = (168133000000000)_!.$$

Using that l is divisible by all prime numbers except 5, one easily sees that in (7) each term on the right beyond the first term is divisible by $2s \cdot 36!$. Calculating F_s modulo $2s \cdot 36!$ by means of the technique explained earlier, and dividing by $2s$, one finds l modulo $36!$:

$$l = (\ldots {}^{2}6{}^{3}351{}^{1}3{}^{1}7{}^{1}1{}^{2}3471{}^{1}604{}^{1}6{}^{1}3{}^{1}8{}^{1}04768{}^{1}0{}^{1}049000120100)_!.$$

One may also compute l directly from (6), if a good method for computing logarithms is available.

Fixed points

The power series expansion also comes in when one wishes to determine the *fixed points* of the Fibonacci sequence, i.e., the numbers s for which $F_s = s$. It is very easy to see that among the ordinary integers the only examples are $F_0 = 0$, $F_1 = 1$, $F_5 = 5$. In $\hat{\mathbf{Z}}$, there are exactly eight additional fixed points, namely the following profinite numbers:

$$\begin{aligned}
z_{1,-5} &= (\ldots 7{}^{1}1548{}^{1}6{}^{1}786{}^{1}065657871411001)_!,\\
z_{1,-1} &= (\ldots {}^{1}8{}^{2}1300813{}^{1}5180733953122001)_!,\\
z_{1,0} &= (\ldots {}^{1}3{}^{1}6507{}^{1}6{}^{1}47{}^{1}168{}^{1}1133471411001)_!,\\
z_{1,5} &= (\ldots {}^{1}9{}^{2}14{}^{1}86{}^{1}6{}^{1}16{}^{1}62{}^{1}1290{}^{1}0071411001)_!,\\
z_{5,-5} &= (\ldots {}^{1}2{}^{1}408206177{}^{1}02{}^{1}048800000021)_!,\\
z_{5,-1} &= (\ldots {}^{2}3{}^{2}3{}^{2}13{}^{1}4{}^{1}18{}^{1}4{}^{1}5{}^{1}4{}^{1}1{}^{1}1{}^{1}124871411021)_!,\\
z_{5,0} &= (\ldots {}^{1}8{}^{1}9041030{}^{1}23{}^{1}28524400000021)_!,\\
z_{5,1} &= (\ldots 52{}^{1}83{}^{1}437{}^{1}10{}^{1}33{}^{1}13{}^{1}10916244021)_!.
\end{aligned}$$

The notation $z_{a,b}$, for $a \in \{1, 5\}$, $b \in \{-5, -1, 0, 1, 5\}$, is chosen because we have

$$z_{a,b} \equiv a \bmod 6^k, \qquad z_{a,b} \equiv b \bmod 5^k$$

for all positive integers k; this uniquely determines $z_{a,b}$ as a fixed point of the Fibonacci sequence. (For $a = b \in \{1, 5\}$ one may take $z_{a,b} = a$.)

There are several techniques that one may use to calculate the numbers $z_{a,b}$ to any required precision. The first is to start from any number x_0 that satisfies $x_0 \equiv a \bmod 24$, $x_0 \equiv b \bmod 5^k$, where k is at least one quarter of the required number of digits, and $k \geq 2$, and next to apply the iteration $x_{i+1} = F_{x_i}$. This converges to $z_{a,b}$ in the required precision, but the convergence is not very fast. One can accelerate this method by choosing a starting value x_0 for which $x_0 - a$ has a greater number of factors 2 and 3. The second method is to apply a Newton iteration to find a zero of the function $F_s - s$:

$$x_{i+1} = x_i - \frac{F_{x_i} - x_i}{lL_{x_i} - 1}.$$

This requires some care with the division that is involved, and one needs to know l to the same precision. However, it converges much faster, even if the starting value x_0 only satisfies $x_0 \equiv a \bmod 24$, $x_0 \equiv b \bmod 25$.

The eight fixed points $z_{a,b}$ have, imprecisely speaking, the tendency to approximately inherit properties of a, b. For example, each of $a = 1$ and $b = 0$ is equal to its own square, and, correspondingly, $z_{1,0}$ is quite close to its own square, in the sense that the nine initial digits are the same:

$$z_{1,0}^2 = (\ldots {}^{1}366{}^{2}0407953{}^{1}02255471411001)_!.$$

Each of $a = 1$, $b = -1$ has square equal to 1, and this is almost true for $z_{1,-1}$:

$$z_{1,-1}^2 = (\ldots {}^{2}2{}^{1}7{}^{1}0{}^{1}0000000000000000000001)_!.$$

Studying the expansions of $z_{1,5}$ and $z_{5,1}$, one discovers that for each i with $4 < i \leq 24$ their ith digits add up to i. This is due to the remarkable relation

$$z_{1,5} + z_{5,1} = (\ldots 000000000000000000000000100)_!,$$

which reflects the equality $5 + 1 = 6 = (100)_!$. Likewise, $5 \cdot 1 = 5 = (21)_!$ is reflected in

$$z_{1,5} \cdot z_{5,1} = (\ldots 0000000000000000000000021)_!.$$

However, greater precision reveals that $z_{1,5} + z_{5,1} \neq 6$ and $z_{1,5} \cdot z_{5,1} \neq 5$:

$$\begin{aligned}
z_{1,5} &= (\ldots 2{}^{2}9{}^{2}6{}^{2}6{}^{2}4{}^{1}63{}^{1}9{}^{2}14{}^{1}86{}^{1}6{}^{1}16{}^{1}62{}^{1}1290{}^{1}0071411001)_!,\\
z_{5,1} &= (\ldots 26323{}^{1}0{}^{2}252{}^{1}83{}^{1}437{}^{1}10{}^{1}33{}^{1}13{}^{1}10916244021)_!,\\
z_{1,5} + z_{5,1} &= (\ldots 55000000000000000000000000000100)_!,\\
z_{1,5} \cdot z_{5,1} &= (\ldots {}^{2}5{}^{2}5000000000000000000000000000021)_!.
\end{aligned}$$

The number $z_{5,-5}$ is the most mysterious of all. By analogy, one suspects its square to be close to $5^2 = (-5)^2 = 25 = (1001)_!$, without being exactly equal to it. Confirming this suspicion requires considerable accuracy:

$$\begin{aligned}
z_{5,-5}^2 = (\ldots & {}^{1}4{}_{6}{}^{1}7{}_{4}{}^{1}1{}_{3}{}^{2}2{}^{2}2000000000000000000000000000000000000\\
& 0000000000000000000000000000000000000000000000000\\
& 0000000000000000000000000000000000000000000000000\\
& 0000000000000000000000000000000000000000000000000\\
& 000000000000000001001)_!.
\end{aligned}$$

Here even the expert may be baffled: given that $z_{5,-5}^2$ is different from 25, is there a good reason for the difference not to show up until after the *two hundredth* digit?