Problemen/UWC

Universitaire Wiskunde Competitie

The Universitaire Wiskunde Competitie (UWC) is a ladder competition for students. Others may participate 'hors concours'. The results can also be found on internet at: http://www.nieuwarchief.nl/uwc

Each issue contains three problems A, B and C. A total of 12 points can be obtained for each problem: 8 for a complete and correct answer, at most 2 points for elegance, and at most 2 points for possible generalisations. To compute the overall score, the totals for each problem are multiplied by a factor 3, 4 and 5, respectively.

The three best contributions will be honoured with a Sessions Prize of respectively 100, 50 and 25 Euro. The points of the winners will be added to their total after multiplication by a factor of respectively 0, 1/2 and 3/4. The highest ranked participant will be given a prize of 100 Euro, after which he/she starts over at the bottom of the ladder with 0 points.

Twice a year there is a Star Problem, of which the editors do not know any solution. Whoever first sends in a correct solution within one year will also receive a prize of 100 Euro.

Group contributions are welcome. Submission by email (in LaTeX) is preferred; participants should repeat their name, address, university and year of study at the beginning of each problem/solution. The submission deadline for this session is August 1, 2005.

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Problem A

A student association organises a large-scale dinner for 128 students. The chairs are numbered 1 through 128. The students are also assigned a number between 1 and 128. As the students come into the room one by one, they must sit at their assigned seat. However, 1 of the students is so drunk that he can't find his seat and takes an arbitrary one. Any sober student who comes in and finds his seat taken also takes an arbitrary one. The drunken student is one of the first 64 students. What is the probability that the last student gets to sit in the chair assigned to him?

Problem B

We consider amino acids A, B and C and proteins formed by an ordered sequence of these. We also consider 9 enzymes which modify the proteins by replacing two adjacent amino acids by two other amino acids. These substitutions are given by:

$$AA \to BC,\ \ AB \to CC,\ \ AC \to BA,$$
$$BA \to CB,\ \ BB \to CA,\ \ BC \to AA,$$
$$CA \to BB,\ \ CB \to AC,\ \ CC \to AB.$$

We define classes of proteins as follows. If a protein has been modified by an enzyme, then it still belongs to the same class of proteins. Two proteins belong to two different classes of proteins if there doesn't exist a set of enzymes that is able to modify one of the proteins into the other.

1. How many classes of proteins consisting of 12 amino acids do there exist,
2. How many proteins belong to the class of proteins of $ABCCBAABCCBA$?

Problem C

In what follows, P stands for the set consisting of all odd prime numbers; M is the set consisting of all natural 2-powers $1, 2, 4, 8, 16, 32, \ldots$; T is the set consisting of all positive integers that can be written as a sum of at least three consecutive natural numbers.

1. Show that the set theoretic union of P,M, and T coincides with the set consisting of all the natural numbers..
2. Show that the sets P,M, and T are pairwise disjoint.
3. Given $b \in T$, determine $t(b)$ in terms of the prime decomposition of b, where by definition $t(b)$ stands for the minimum of all those numbers $t > 2$ for which b admits an expression as sum of t consecutive natural numbers.
4. Consider the cardinality $C(b)$ of the set of all odd positive divisors of some element b of T. Now think of expressing this b in all possible ways as a sum of at least three consecutive natural numbers. Suppose this can be done in $S(b)$ ways. Determine the numerical connection between the numbers $C(b)$ and $S(b)$.

Edition 2004/4

Op de ronde 2004/4 van de Universitaire Wiskunde Competitie ontvingen we inzendingen van Ferry Kwakkel, Peter Bruin, Jaap Spies and Jan van Lune, Leen Bleijenga, Nicky Hekster and Michiel Vermeulen, Hendrik Hubrechts.

Problem 2004/4-A

1. Show that there exist infinitely many $n \in \mathbf{N}$, such that $S_n = 1 + 2 + \ldots + n$ is a square.
2. Let $a_1, a_2, a_3, \ldots$ be those squares. Calculate $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

Solution This problem has been solved by Ferry Kwakkel, Peter Bruin, Jaap Spies and Jan van Lune. The solution below is based on the solution of Jan van Lune.
We note that $S_n = \frac{1}{2}n(n+1)$, hence we have to solve $\frac{1}{2}n(n+1) = y^2$. Multiplying both sides by 8, we get $4n^2 + 4n = 8y^2$. Substituting x for $2n+1$ gives the Pell-equation $x^2 - 1 = 8y^2$ or $x^2 - 8y^2 = 1$ or $(x + \sqrt{8}y) \cdot (x - \sqrt{8}y) = 1$. It can be proved that the solutions of $x^2 - 8y^2 = 1$ are given by $x_n + \sqrt{8}y_n = (3 + \sqrt{8})^n$ (using the theory of continued fractions of algebraic numbers of degree 2). The solutions of $x^2 - 8y^2 = 1$ are given by

$$(x_n, y_n) = (1,0), (3,1), (17,6), (99,35), (577,204), (3363,1189), (19601,6930), \ldots$$

For the solutions (x_n, y_n) we can deduce the relation between y_{n-1}, y_n, y_{n+1}: $y_{n+1} = 6y_n - y_{n-1}$. Therefore $y_{n+1}/y_n = 6 - y_n/y_{n-1}$. The theory of continued fractions tells us that $\lim_{n\to\infty} y_{n+1}/y_n = q$ exists. We can show that q is the largest root of $z^2 - 6z + 1$. Therefore $q = 3 + 2\sqrt{2}$. Finally

$$\lim_{n\to\infty} a_{n+1}/a_n = \lim_{n\to\infty} \left(y_{n+1}/y_n\right)^2 = 17 + 12\sqrt{2}.$$

See *An introduction to the theory of numbers*, G.H. Hardy and E.M. Wright, Chapter 14, Section 5.

Problem 2004/4-B Let G be a finite set of elements and $\cdot$ a binary associative operation on G. There is a neutral element in G and that is the only element in G with the property $a \cdot a = a$. Show that G with the operation $\cdot$ is a group.

Solution This problem has been solved by Ferry Kwakkel, Peter Bruin, Leen Bleijenga, Jaap Spies, Nicky Hekster and Michiel Vermeulen. Michiel Vermeulen's solution is given here.
Notice that the finiteness of G is essential. If G were infinite then the problem could not be solved. Let for instance $G = \mathbf{N}$ and let the operation $\cdot$ be the multiplication. Then 1 is the unique neutral element, however $(\mathbf{N}, \cdot)$ is not a group.
Almost all solutions sent in included the implication that $x^i = x^j$ implies $x^{j-i} = e$, the neutral element. In the description of the problem the only thing which was said was that the neutral element e is the unique element for which $e \cdot e = e$. So for arbitrary x

we have to find an integer $m > 0$ such that $x^m \cdot x^m = x^m$. Then x^m is the unique neutral element and x^{m-1} is the inverse of x.
A set G together with an operation $\cdot$ is a group when the group axioms are satisfied:
1. Closure: if $x, y \in G$, then $x \cdot y \in G$.
2. Associativity: if $x, y, z \in G$, then $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.
3. Identity element: there exists an identity element e such that $x \cdot e = e \cdot x = x$ for all $x \in G$.
4. Inverse: for any $x \in G$ there exists a $y \in G$ such that $x \cdot y = y \cdot x = e$.

G satisfies the first two conditions. Furthermore the element for which $e \cdot e = e$ is unique. We have to prove for any $x \in G$ that $x \cdot e = e \cdot x = x$ and that x has an inverse.
Let $x \in G$ and consider the sequence $x, x^2, x^3, \ldots$. Since G is finite there exist integers i and j such that $x^i = x^j$. Define $t = j - i$.

Lemma. *$\forall k > 0$ we have $x^{i+kt} = x^i$.*

Bewijs. This lemma can be proved by induction. For $k = 1$ we find $x^i = x^j$ and if we proved the lemma for some k then for $k + 1$ we have $x^{i+(k+1)t} = x^{(i+kt)+t} = x^{i+t} = x^i$.
Now we are able to construct the neutral element. Choose k such large that $kt - i > 0$. Since $x^{i+kt} = x^i$ we find (by multiplying both sides by x^{i+kt}) that $\left(x^{kt}\right)^2 = x^{kt}$. Hence x^{kt} is the neutral element and x^{kt-1} is the inverse of x. □

See *Topics in algebra*, I.N. Herstein, Chapter 2, Definition 1.

Problem 2004/4-C Let $\{a_n\}_n$ be a sequence ($n \geq 0$), with $a_n \in \{\pm 1\}$ for all n. Define

$$S_n = \sum_{k=0}^{n} a_k a_{n-k}.$$

Prove that $\exists C > 0 : \forall m > 0 : \exists n > m : |S_n| > C\sqrt{n}$.

Solution We did not receive any solutions to this problem. We present here the solution by Jaap Korevaar, who proposed the problem. We give his solution here.
For $0 < r < 1$ define the function $A(r, z)$ to be

$$A(r, z) = \sum_{k=0}^{n} a_k e^{ikz} r^k.$$

Moreover, let

$$B(r, z) = A(r, z)^2 = \sum_{k=0}^{n} S_k e^{ikz} r^k.$$

Then clearly

$$\frac{1}{1 - r^2} = \sum_{k=0}^{\infty} a_k^2 r^{2k} = \frac{1}{2\pi} \int_{\mathbf{T}} A(r, z)\overline{A(r, z)}|dz|.$$

Here $\mathbf{T}$ is the unit circle in the complex plane, and integration is done clockwise. Using Cauchy-Schwarz, we see that

$$\left(\frac{1}{2\pi} \int_{\mathbf{T}} A(r, z)\overline{A(r, z)}|dz|\right)^2 \leq \frac{1}{2\pi} \int_{\mathbf{T}} B(r, z)\overline{B(r, z)}|dz| \frac{1}{2\pi} \int_{\mathbf{T}} 1|dz|.$$

The last integral is 1, and the second to last is equal to

$$\frac{1}{2\pi} \int_{\mathbf{T}} B(r, z)\overline{B(r, z)}|dz| = \sum_{k=0}^{\infty} S_k^2 r^{2k}.$$

We therefore see that

$$\left(\frac{1}{1 - r^2}\right)^2 \leq \sum_{k=0}^{\infty} S_k^2 r^{2k} \tag{1}$$

for $0 < k < 1$. If from a certain point on all S_k are smaller than $\sqrt{k}/2$, then the right-hand side becomes

$$\sum_{k=0}^{\infty} S_k^2 r^{2k} \le D + \sum_{k=0}^{\infty} \frac{k}{4} r^{4k} = D + \frac{r^2}{4}\left(\frac{1}{1-r^2}\right)^2.$$

Here D is defined by

$$D = \sum_{k=0}^{\infty} \max(0, S_k^2 - k/4).$$

Note that this is a finite sum.
For r close enough to 1, we see that the term D on the right becomes negligible with respect to the other term. It follows that (1) cannot be satisfied. This contradiction proves the desired statement.

Results of Session 2004/4

	Name	A	B	C	*Total*
1.	Peter Bruin	8	8	-	56
2.	Ferry Kwakkel	7	4	-	37

Final Table after Session 2004/4
We give the top 3, the complete table can be found on the UWC website.

	Name	*Points*
1.	Gerben Stavenga e.a.	136
2.	Filip Cools e.a.	123
3.	Peter Bruin	99