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Beroemde problemen

The Invariant Subspace Problem

Heeft elke begrensde lineaire operator, werkend op een Hilbert ruimte, een niet-triviale invariante deelruimte? Het antwoord is positief voor zowel eindig-dimensionale ruimtes als voor niet-separabele ruimtes. Het onopgeloste probleem voor het geval daar tussenin, dus voor separabele Hilbert ruimtes staat bekend als het invariante deelruimte probleem.

Professor B.S. Yadav van de Indian Society for History of Mathematics in Delhi heeft de afgelopen 25 jaar veel over het onderwerp gepubliceerd en geeft hier een historisch overzicht.

The invariant subspace problem is the simple question: “Does every bounded operator T on a separable Hilbert space H over $\mathbf{C}$ have a non-trivial invariant subspace?” Here non-trivial subspace means a closed subspace of H different from $\{0\}$ and different from H. Invariant means that the operator T maps it to itself. The problem is easy to state, however, it is still open. The answer is ‘no’ in general for (separable) complex Banach spaces. For certain classes of bounded linear operators on complex Hilbert spaces, the problem has an affirmative answer.

It seems unknown who first stated the problem. It apparently arose after Beurling [1] published his fundamental paper in Acta Mathematica in 1949 on invariant subspaces of simple shifts, or after von Neumann's unpublished result on compact operators which we shall discuss in the sequel.

A history of the problem

Let H be any complex Hilbert space and T a bounded operator on H. An eigenvalue λ of T clearly yields an invariant subspace of T, namely the kernel of $T-\lambda$. So if T has an eigenvalue, the problem is solved (the special case where T is multiplication by λ being trivial). However, not every bounded operator T on a complex Hilbert space has an eigenvalue. For example, the shift operator T on ℓ^2, the Hilbert space of all square-summable sequences of complex numbers, defined by

$$Tx = (0, x_0, x_1, \ldots)$$

for each vector $x = (x_0, x_1, \ldots) \in \ell^2$, does not have any eigenvalue. However, if H is finite-dimensional, then of course every T on H has an eigenvalue, so the problem is solved for finite dimensional complex vector spaces.

Next, suppose H is infinite-dimensional but not separable. Let T be a bounded operator on H. Take a non-zero vector x and consider the closed subspace M generated by the vectors $\{x, Tx, T^2x, \ldots\}$. Then M is invariant under T and obviously $M \neq \{0\}$. Moreover, M does not coincide with H as this would contradict that H is non-separable. Thus every operator T on a non-separable infinite-dimensional complex Hilbert space H has a non-trivial invariant subspace.

What remains to be examined is actually the invariant subspace problem: does every bounded operator T on an infinite-dimensional separable complex Hilbert space H have a non-trivial invariant subspace?

The solution for Banach spaces

During the annual meeting of the American Mathematical Society in Toronto in 1976, the young Swedish mathematician Per Enflo announced the existence of a Banach space and a bounded linear operator on it without any non-trivial invariant subspace. Enflo was visiting the University of California at Berkeley at that time. However, nothing appeared in print for several years and it was only in

1981 that he finally submitted a paper for publication in Acta Mathematica. Unfortunately the paper remained unrefereed with the referees for more than five years, though its manuscript had a world-wide circulation amongst mathematicians. This happened, as they say, because the paper was quite difficult and not well written. The paper was ultimately accepted in 1985 and it actually appeared in 1987 with only minor changes: [4]. However, he had announced his construction of the counterexample earlier in the "Seminaire Maurey-Schwarz (1975–76)" and subsequently in the "Institute Mittag-Leffler Report 9 (1980)"; see [2], [3].

In the meantime, C.J. Read, following the ideas of Enflo, also constructed a counterexample and submitted it for publication in the Bulletin of the London Mathematical Society. The paper was quickly refereed and it appeared in July 1984 [5] breaking the queue of backlog for publication. A shorter version of this proof was published again by Read in 1986. He also constructed in 1985 [6] a bounded linear operator on the Banach space ℓ^1 without non-trivial invariant subspaces.

The temptation on the part of Read to have precedence over Enflo for solving the problem was considered professionally unethical by many mathematicians. Particularly, because his work was essentially based on ideas of Enflo. For example, the French mathematician Bernard Beauzamy also sharpened the techniques of Enflo and produced a counterexample. He presented it at the Functional Analysis Seminar, University of Paris (VI-VII) in February, 1984. But he declined to publish his result in the Bulletin of the London Mathematical Society, although the Editors offered him the same facilities as they did to Read. Beauzamy's paper appeared later in June 1985 in Integral Equations and Operator Theory.

The ℓ^1-example of [6] was further simplified by A.M. Davie, as can be found in Beauzamy's book (1988).

One should not get the impression that all counterexamples which have been produced so far are based directly or indirectly on the techniques developed by Enflo. As a matter of fact, a series of papers written by Read himself after his first paper in 1984 makes a further significant contribution to the subject. For example, the counterexample that he constructed on ℓ^1 in 1985 is characteristically different from and simpler than Enflo's, and could be counted as a major achievement. Again, in yet another paper in 1988, Read constructed a bounded linear operator on ℓ^1 which has no invariant closed sets (let alone invariant subspaces) other than the trivial ones. Not only is this a stronger result, it also gives rise to a new situation: suppose that the invariant subspace problem is solved in the negative one day (as in the case of Banach spaces), one would ask a next question: "Does every bounded operator have a non-trivial invariant closed set?"

Building on his earlier work, Read published in 1997 an example of a quasinilpotent bounded operator (i.e., $\lim ||T^n||^{1/n} = 0$) on a Banach space without a non-trivial invariant subspace. The same result is nicely described in [8].

Von Neumann's unpublished result

John von Neumann (unpublished) showed that every compact operator on a Hilbert space has a non-trivial invariant subspace. The first proof of this result was published by Aronszajn and Smith in 1954. The result was extended to polynomially compact operators by A.R. Bernstein and A. Robinson in 1966 using techniques from non-standard analysis due to Robinson. Halmos translated their proof into standard analysis. Interestingly, his paper appeared in the same issue of Pacific Journal of Mathematics, just after theirs. In 1967, Arveson and Feldman transformed the result in a still more general form by essentially chiselling the technique of Halmos: if T is a quasinilpotent operator such that the uniformly closed algebra generated by T contains a non-zero compact operator, then T has a non-trivial invariant subspace.

Per Enflo

The Lomonosov technique

The result of Arveson and Feldman was, in a sense, the climax of the line of action initiated by von Neumann. However, operator theorists were stunned in 1973 when the young Russian mathematician V. Lomonosov obtained a more general result:

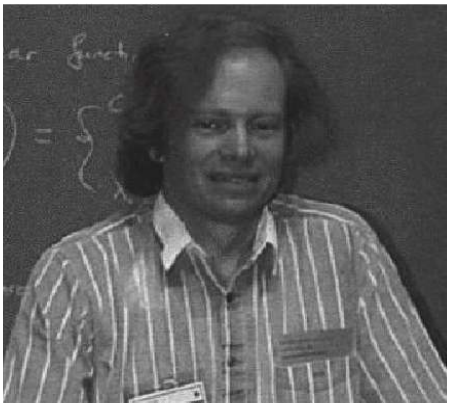
If a non-scalar bounded operator T on a Banach space commutes with a non-zero compact operator, then T has a non-trivial hyper-

Cyclic vectors

A vector x in H is called a cyclic vector of a bounded operator T on H if the closure of the span of all $T^n x$ equals H. The operator T has no non-trivial invariant subspaces if and only if every non-zero vector is a cyclic vector of T: if a vector x is non-cyclic, then the closure of the span of all $T^n x$ is a non-trivial invariant subspace of T. And if M is a non-trivial invariant subspace, then every non-zero vector in M is non-cyclic.

Normed linear spaces

A vector space X over the field $\mathbf{R}(\mathbf{C})$ of real (complex) numbers is called a normed linear space if each vector $x \in X$ has a 'norm' $||x|| \in \mathbf{R}$, such that $||x|| \geq 0$ and $||x|| = 0$ if and only if $x = 0$, $||\alpha x|| = |\alpha|\,||x||$ for each scalar α and $||x + y|| \leq ||x|| + ||y||$ for all x, y in X. Every normed linear space X is a metric space with the metric defined by $d(x, y) = ||x - y||$. A Banach space is a normed linear space which is complete (as a metric space). A Hilbert space is a Banach space endowed with the additional structure of an inner product $< x, y >$ such that the norm is related to the inner product by the equality $< x, x >= ||x||^2$. By a bounded operator on a Banach space X one means a linear transformation of X to itself such that there exists a constant $K > 0$ for which $||Tx|| \leq K||x||$ for all $x \in X$. The operator norm of a bounded operator T, denoted $||T||$, is by definition $||T|| := \sup\{||Tx||/||x|| \;;\; x \neq 0\}$. A normed space X is called separable if it has a countable dense subset.



C.J. Read

invariant subspace (this means, a subspace which is invariant under every operator that commutes with T).

This theorem was quite exciting for many reasons:

I. Lomonosov used a brand-new technique (namely, an ingenious use of Schauder's fixed point theorem), entirely different from the line of action followed hitherto by other mathematicians.

II. His result was much stronger than what was known so far: every polynomially compact operator has a non-trivial invariant subspace.

III. His theorem highlighted another, stronger, form of the 'invariant subspace problem': "Does every bounded linear operator on a Hilbert space have a non-trivial hyperinvariant subspace?"

IV. Many mathematician tried to find alternative proofs of Lomonosov's theorem, say, by replacing the use of Schauder's fixed-point theorem by the Banach contraction principle, but the theorem stands as it was even today. M. Hilden, however, succeeded in proving its special case that every non-zero compact operator has a non-trivial hyperinvariant subspace without using any fixed-point theorem.

In fact, Hilden assumed without any loss of generality a non-zero compact operator also to be quasinilpotent: if a non-zero compact operator is not quasinilpotent, then it must have a non-zero eigenvalue, and hence the eigenspace corresponding to this eigenvalue is a non-trivial hyperinvariant subspace. Hilden exploits the quasinilpotence of the compact operator to finish his proof.

V. Initially it was felt that Lomonosov's theorem might lead to a solution of the general 'invariant subspace problem' in the affirmative. However, seven years after his result, in 1980, Hadvin-Nordgren-Radjavi-Rosenthal gave an example of an operator that does not commute with any non-zero compact operator.

VI. A number of extensions and applications of Lomonosov's theorem have been obtained by several mathematicians.

Normal-like non-normal operators

A bounded operator T on a Hilbert space H is called 'normal' if it commutes with its adjoint T^*. It is called 'subnormal' if it is the restriction of a normal operator to an invariant subspace, and 'hyponormal' if $||T^*x|| \leq ||Tx||$ for all $x \in H$. It is not difficult to see that normality $\Rightarrow$ subnormality $\Rightarrow$ hyponormality, but the converse is true in neither case. An important result in operator theory, known as Fuglede's theorem, states that if T is a normal operator and $S \in B(H)$ is such that $TS = ST$, then $T^*S = ST^*$.

Fuglede's theorem implies that every non-scalar normal operator on a Hilbert space has a non-trivial hyperinvariant subspace. To show the existence of non-trivial invariant (hyperinvariant) subspaces of non-normal operators satisfying certain nice conditions has been a fascinating subject for operator theorists. One of the most striking results in this direction was due to Scot Brown who showed in 1978 that every subnormal operator has a non-trivial invariant subspace. J.E. Thomson (1986) found a simple and elegant proof of Brown's result. Consider the Hilbert space $L^2(\mu)$, where μ is a suitable positive Borel measure with compact support in the complex plane. Thomson makes a decisive use of the fact that a cyclic subnormal operator can be modelled as a multiplication by z on the closure of the space of all polynomials in $L^2(\mu)$. (A bounded operator T on the Hilbert space H is called cyclic if there exists $x \in H$ such that the closure of the span of $\{T^n_x;\ n \geq 0\}$ equals H.) As a matter of fact, Thomson's method gives rise to a more general result:

Let A be a subalgebra of $L^\infty(\mu)$ containing z and let H be a subspace of $L^2(\mu)$. If H contains constants and is invariant for A, then there is a non-trivial subspace of H that is A-invariant.

In 1987 Brown, extending his techniques and using descriptions of hyponormal operators due to M. Putinar (1984), proved that every hyponormal operator with the spectrum having a non-empty interior has a non-trivial invariant subspace.

Lastly we mention yet another significant result in this direction due to Brown, Chevreau and Pearcy: every contraction whose spectrum contains the unit circle has a non-trivial invariant subspace.

Heritages of the problem

For an operator T one denotes by Lat$T$ the lattice of all invariant subspaces of T, with set-inclusion as partial order. For a general operator T, it is extremely difficult to describe Lat$T$, particularly when we do not know whether there exists a bounded operator T for which Lat$T$ is isomorphic to the lattice $\{0, 1\}$ (this is the invariant subspace problem!). However, for certain special operators T, namely the shifts and the Volterra operators, the structure of Lat$T$ is completely known. We now describe this, and discuss the role of shifts and their invariant subspaces in the structure theory of operators, as initiated by G.-C. Rota.

Let $\{e_n\}_{n=0}^\infty$ be an orthonormal basis for H. The operator U on H such that $Ue_n = e_{n+1}, n = 0, 1, 2\ldots$ is called the (forward) shift operator. A simple calculation shows that its adjoint S is the backward shift, given by $Se_0 = 0$ and $Se_n = e_{n-1}$ for $n \geq 1$. We shall be concerned with

Compact operators

An operator T on a Banach space X is called 'compact' (completely continuous) if for every bounded subset $A \subset X$, the closure $\overline{T(A)}$ of its image is compact in X. An operator T is called 'polynomially compact' if there exists a polynomial p such that the operator $p(T)$ is compact. Every compact operator is obviously polynomially compact, but the converse is not true; examples can be found in Paul Halmos' *A Hilbert space problem book* (1967).

A bounded operator T is 'quasinilpotent' if $\lim_{n\to\infty} ||T^n||^{1/n} = 0$.

We say that a subalgebra of the algebra of bounded operators on X is *uniformly closed* if it is closed with respect to the operator norm.

the following concrete representations of U and S.

Let $L^2 = L^2(C, \mu)$ be the Hilbert space of all square-integrable functions defined on the unit circle C, where μ is the normalized Lebesgue measure on C(i.e.$\mu(C) = 1$). If for each integer $n, e_n = e_n(z) = z^n$, then $\{e_n\}_{n=-\infty}^{\infty}$ is an orthonormal basis of L^2. The Hardy space H^2 is the closed subspace of L^2 generated by the vectors $\{e_0, e_1, e_2, \ldots\}$. We see that the multiplication by $e_1(z) = z$ on H^2 is U.

As a second example, let ℓ^2 be the Hilbert space of all square-summable complex sequences $x = (x_n)_{n=0}^{\infty}$. Then U and S on ℓ^2 appear as $Ux = (0, x_0, x_1, \ldots)$ and $Sx = (x_1, x_2, x_3, \ldots)$.

Beurling's theorem and its ramifications

In 1949, A. Beurling characterized the invariant subspaces of the shift operator on the Hardy space H^2 on the unit circle. His result is:

If M is an invariant subspace of the shift operator on the Hardy space H^2 on the unit circle C, then there exists an inner function ϕ on C (this means that ϕ is measurable and $|\phi(z)| = 1$ almost everywhere on C), such that $M = \phi H^2$.

If both ϕ_1 and ϕ_2 are such functions, then ϕ_1/ϕ_2 is equal to a constant function almost everywhere.

As Beurling's theorem showed an interplay between the theory of functions and the operator theory, it has naturally had numerous ramifications both in harmonic analysis and functional analysis. Mainly there have been three directions:

I. Replacing the Hardy space of scalar-valued functions by the Hardy space of vector-valued functions;
II. Extending Beurling's characterization to the Hardy space of scalar-valued functions on the torus;
III. Viewing (i) and (ii) in the sense of de Branges, which puts Beurling's theorem as well as its vector-valued generalizations due to Halmos (1961) and others in a more general setting.

John von Neumann (1903–1957)

Weighted shifts

Shifts form an important class of operators. They have been rightly called the 'Building Blocks' of operator theory. Many important operators are, in a sense, 'made up' of shifts, for example, every pure isometry is a direct sum of shifts and every contraction with powers strongly tending to zero is a 'part' of a backward shift.

More importantly, shifts serve as an unending source of counterexamples. Read uses a shift to construct his counterexample of a bounded operator on a Banach space without a non-trivial invariant subspace.

Let H be a Hilbert space with an orthonormal basis $\{e_n\}_{n=0}^{\infty}$ and let $w = \{w_n\}_{n=1}^{\infty}$ be a sequence of non-zero complex numbers. Consider the weighted forward shift T_w:

$$T_w e_n = w_{n+1} e_{n+1}, \quad n = 0, 1, 2, \ldots$$

and the corresponding weighted backward shift S_w:

$$S_w e_0 = 0,$$
$$S_w e_n = \bar{w}_n e_{n-1}, \quad n = 1, 2, 3, \ldots$$

A weighted forward shift is the adjoint of a weighted backward shift and vice-versa. Note that a subspace M is invariant under an operator T if and only if its orthogonal complement $M^{\perp}$ is invariant under T^*. Hence determining LatS_w is equivalent to determining LatT_w.

Let M_n denote the closed subspace spanned by

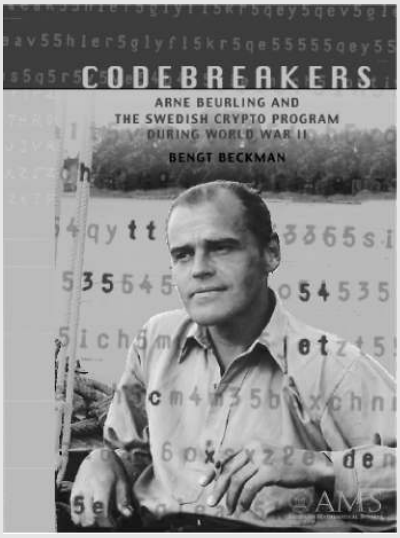
$$\{e_0, e_1, \ldots e_n\}.$$

Then $M_n \in$ LatS_w for all n. Under certain conditions on the weight sequence w, one can show that LatS_w consists of M_n's only: if a weight sequence $w = \{w_n\}_{n=1}^{\infty}$ is such that $\{|w_n|\}$ is monotonically decreasing and

$$\sum_{n=0}^{\infty} |w_n|^2 < \infty,$$

then every non-trivial invariant subspace in LatS_w is some M_n.

This result is due to N.K. Nikolskii (1965). The case $w_n = 2^{-n}$ was obtained in 1957 by W.F. Donoghue.



The work of the Swedish mathematician Arne Beurling (1905–1986) has been pace-setting in many directions in abstract harmonic analysis, functional analysis and operator theory. When conscripted in 1931, he reconstructed Swedish cryptology ingeniously leading to information vital for the survival of Sweden during the World War II. Although appointed professor at the Institute of Advanced Study, Princeton in 1954, he always missed the right social environment and would not even apply for a Green Card. A review of this book appeared in September 2003 in the Notices of the AMS. ***Codebreakers: Arne Beurling and the Swedish Crypto Program during World War II***, Bengt Beckman, AMS 2002, ISBN 0-8218-2889-4.

Volterra integral operators

Consider the Volterra integral operator V defined on $L^2(0,1)$ by

$$(Vf)(x) = \int_0^x f(t)dt, \qquad 0 \le x \le 1,$$

for all $f \in L^2(0,1)$. This operator is another one whose invariant subspaces have been characterized. For each $\alpha \in [0,1]$, let

$$M_\alpha = \{f \in L^2(0,1) : f = 0 \text{ almost everywhere on} [0,\alpha]\}.$$

Obviously, $M_\alpha \in \mathrm{Lat}V$ for all $\alpha \in [0,1]$. In fact,

$$\mathrm{Lat}V = \{M_\alpha : \alpha \in [0,1]\}.$$

This was proven by J. Dixmier (1949) in case of the real space $L^2(0,1)$. W.F. Donoghue and M.S. Brodskii independently settled it in 1957 for the complex space $L^2(0,1)$.

These results have been extended to integral operators K on $L^2(0,1)$ defined by

$$(Kf)(x) = \int_0^x k(x,y)f(y)dy, \quad 0 \le x \le 1,$$

for all $f \in L^2(0,1)$, where $k(x,y)$ is a square-integrable function on $[0,1] \times [0,1]$. The characterization of $\mathrm{Lat}K$ in this case may be used to obtain a functional-analytic proof of the famous classical Titchmarsh convolution theorem (G.K. Kalisch, 1962).

Rota's models of linear operators

By a part of an operator T on a Hilbert space H, we mean the restriction $T|_M$ of T to an invariant subspace M of T.

Let $l^2(H)$ denote the Hilbert space of all square-summable sequences $x = (x_0, x_1, \ldots, x_n, \ldots)$ in H. Take a bounded sequence $w = (w_n)$ of positive real numbers. The backward shift S_w on $l^2(H)$ is given by

$$S_w x = (w_1 x_1, w_2 x_2, \ldots, w_{n+1} x_{n+1}, \ldots).$$

Put $\beta_0 = 1$ and $\beta_n = w_1 w_2 \cdot \ldots \cdot w_n$, for $n \ge 1$. One has the following result.

Suppose T is a bounded operator on H and $\sum_{n=0}^{\infty} \beta_n^{-2} ||T^n||^2 < \infty$.

Then T is similar to a part of S_w on $l^2(H)$ in the following sense: define $A : H \to l^2(H)$ by $Ax = \{\beta_0^{-1}x, \beta_1^{-1}Tx, \beta_2^{-1}T^2x, \ldots\}$, then the image M of A is closed and $S_w A = AT$.

This implies M is an invariant subspace of S_w and T is similar to $S_w|_M$.

If the spectral radius,

$$r(T) := \lim_{n\to\infty} \left(||T^n||^2\right)^{1/n},$$

of T is less than 1, then the conditions of the above result are satisfied for the constant sequence $w_n = 1$. This observation leads to the result of G.-C. Rota (1960):

If a bounded operator T on a Hilbert space H has spectral radius $r(T) < 1$, then T is similar to a part of the standard backward shift on $l^2(H)$. In particular, this holds for a strict contraction T, i.e. if $||T|| < 1$.

Since any bounded operator can be 'scaled' so as to be a strict contraction, Rota's work yields a reformulation of the invariant subspace problem: *Are the minimal non-zero invariant subspaces of backward shifts one-dimensional?*

More details on this work initiated by Rota may be found in [7].

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