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Onderwijs

A significant amount of algebra

"Als wij het nog abstracter maken komen er nog minder wiskundestudenten." Deze uitspraak over het wiskundeonderwijs komt van een hoge medewerker van het ministerie Onderwijs, Cultuur en Wetenschappen. Diametraal hierop is het standpunt van Zalman Usiskin, professor voor didactiek en directeur van het University of Chicago School Mathematics Project (UCSMP). Volgens hem zijn mensen juist door het leren van algebra en andere geavanceerdere wiskunde beter in staat om de wereld te begrijpen en om hun werk te kunnen doen. Zalman Usiskin gaf in zijn slotlezing op de Nationale Wiskunde Dagen 2004 vele goede redenen voor onderwijs van algebra op de middelbare school. Usiskin levert hiermee een bijdrage aan de discussie over het algebraonderwijs in Nederland die momenteel in volle gang is.

It is a great pleasure to have been invited to speak in the Netherlands for the first time. Over the years, I have been influenced by the work of Dutch mathematics educators on a number of occasions. My doctoral dissertation involved the testing of a geometry course for average 10th grade students in which the geometry was developed through reflections and other geometric transformations. The development in the geometry book that Art Coxford and I wrote was influenced quite a bit by a set of schoolbooks written in Dutch by Rudolf Troelstra and others in the 1960s. But even more relevant to this conference are the connections I had with Hans Freudenthal. I first met Freudenthal in the 1970s when he visited the University of Chicago. Beginning in 1979, I did some research on the theory of learning of Dina and Pierre Marie van Hiele, both students of his. Then in 1980 my wife and I had dinner with him, just the three of us, during the International Congress on Mathematical Education meeting. Freudenthal was such a major figure and this was quite a special event for me. A few years later I was asked to review his book *The Didactical Phenomenology of Mathematical Structures*, and I was pleased and surprised to see that the fundamental conceptualization Freudenthal presented in that book helped me in understanding many of the ideas and some of the work that I had been doing over the preceding decade. It is an honor to have been asked to speak under the auspices of the institute that bears his name.

I was asked to give a talk that would be of interest both to secondary school teachers and research mathematicians, that would strike a balance between mathematical content and didactical and curricular issues, that would involve policy, that would inspire you, and that would cause you to leave this conference excited by what you have heard. That would be difficult enough, but my task is made more difficult by my ignorance of what you know, what examples you have seen and exactly what you teach. This is a talk about curriculum policy, and so I will try to give arguments that you could use with those who do not teach mathematics, with parents, and with others outside the mathematics community who question what we do and why we do it. I apologize in advance if what I say is obvious; I can only tell you that it would not be obvious to secondary school teachers in the United States.

Introduction

The algebra we teach to students can be traced back through the Greek mathematician Diophantus to the Babylonians over 3500 years ago, but the language we use in today's algebra is relatively new as mathematics goes, dating back only to the French mathematician Viète in 1591 and the systematization of the content by Euler in 1770. In the United States as recently as 1910, less than

15% of the age cohort entered high school and only they studied any algebra [1]. I estimate that 1 in 20 people studied a second year of algebra.

Through the 20th century the situation changed dramatically. By 1972, virtually all students entered high school and about 72% of them were taking one year of algebra. About half of these students took a second year of algebra. Algebra was still viewed as a high school subject and introduced before 9th grade only to perhaps 10% of students. Yet by 1998, algebra was being introduced to virtually all students before high school, over 90% of students were taking one year of algebra by the end of the 9th grade (25% by the end of 8th grade), and 2/3 of students were studying a second year of algebra in 10th or 11th grade [2]. In the first year of algebra they are taught linear equations and inequalities, graphing of lines, operations with polynomial expressions (including factoring), laws of positive integer powers, linear systems, square roots, and simple quadratics and maybe rational expressions. In the second year, they are most likely to begin with a review of first-year algebra and then be taught rational powers and nth roots, operations with matrices, general notions of functions, linear and quadratic functions, logarithms and exponential functions, sequences, and perhaps some combinatorics and probability [3]. when I speak of “a significant amount of algebra”, I mean this amount.

Notice that I say that students are *taught* these subjects. I did not say that they *learn* them. Many colleges require these subjects but many students have to take remedial algebra courses in college. There is a movement in the U.S. for all students, including those not planning to go to college, including even those who might not finish high school, to take two years of algebra. And I understand you have a minister who wants to decrease the amount of mathematics for all students. Which is the wiser strategy? We may think the answer is obvious, but we are in mathematics! We are biased! Let us try to look in an unbiased manner at the current situation regarding algebra in our world.

Arithmetic is everywhere, algebra is hidden.
Most people realize that they need to know arithmetic. Whole numbers, fractions, decimals, and percents are everywhere. Just pick up a newspaper or magazine, open to any page at random, and count the numbers on it. I have examined the uses of numbers in newspapers around the world and in almost every country, a daily newspaper has a *median* of about 125 numbers on its pages.

Algebra is different. Scan a daily newspaper in Dutch or English and you are not likely to see any algebraic formulas. Adults may be handicapped by a lack of knowledge of arithmetic, but lack of knowledge of algebra does not seem too debilitating. Items for sale in stores have lots of numbers on them... and no algebra. Even scientific publications meant for the educated public and with many numbers in them tend to have little if any algebra.

Yet algebra must be important, for if algebra were not important, why would we require all students to study it? To say “You need algebra for college”, or “You won’t do well on some exams without it” is true. But these reasons do not tell us why algebra is so important that it has become a required element in the mathematics curriculum of almost every secondary school student in the world.

When people are required to learn a subject that they feel they do not need, will not use, and that requires work to learn, there are predictable results. Many of them will dislike the subject and they will be proud of their dislike. No one makes fun of reading. No one makes fun of arithmetic. But people make fun of algebra. My favorite cartoon of this type (shown on the next page) was drawn by a high school student, Brad Haak, whose mother is a mathematics teacher.

This attitude towards algebra is passed on from generation to generation.

The reasons for algebra
There are practical reasons for learning algebra.

Algebra is a gatekeeper
In the United States, without a knowledge of algebra, a person cannot go to any college that has any selectivity. A person is kept from doing many jobs or even entering many job-training programs. Algebra is such a significant gatekeeper that at least one educator has called the study of algebra a *civil right.*

Again, however, knowing that algebra is a gatekeeper begs the question. Why is algebra considered so important that its study is required? I offer some reasons intrinsic to algebra.

Algebra is the language of generalization
Here is the rule for multiplication of fractions, in English: To multiply two fractions, multiply their numerators to get the numerator of the product, and then multiply their denominators to get the denominator of the product. For example,

$$\left(\frac{2}{3}\right)\left(\frac{4}{5}\right)=\frac{(2)(4)}{(3)(5)}=\frac{8}{15}$$

What is the rule in the language of algebra?

$$\left(\frac{a}{b}\right)\left(\frac{c}{d}\right)=\frac{(a)(c)}{(b)(d)}=\frac{ac}{bd}$$

Not only is the language of algebra shorter than the English (or the Dutch), but it also looks like the arithmetic. For this reason, the language of algebra is *easier* than everyday language.

As we all know, there are formulas for area as simple as $A = LW$ (in a rectangle, area equals length times width) that come in handy if any person is looking for a place to live and wants to know how much floor space or wall space there is, or if a person is sewing clothes and wants to determine the amount of material that is needed. There are formulas for perimeter that tell how long a fence a person might need for a field or how much ribbon to tie a package. There are all sorts of formulas in sports, from the calculation of winning percentage to the probability that a team will win any number of games in a row. Income tax, sales taxes, discounts, and virtually every money matter involve applying some formula. People can get along without the formulas — most people do — but they are less likely to be fooled by someone misinterpreting them if they themselves can work with the algebra.

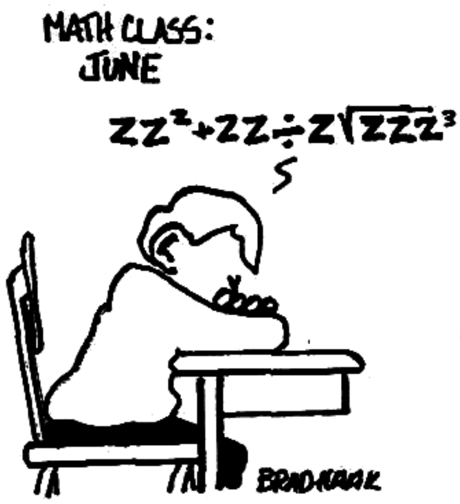
Algebra enables a person to answer all the questions of a particular type at one time
Suppose you have a date of some event in history (in the Gregorian calendar) and you want to know the day of the week that it occurred. You can figure the day out by working from today’s day and date back, accounting for leap years — trying to remember which leap years are the exceptions — and you will get your answer. But if you have many questions of this type, then you want a formula.

One such formula is

$$\begin{aligned} W = {} & d + 2m + \left\lfloor \frac{3(m+1)}{5} \right\rfloor + \\ & + y + \left\lfloor \frac{y}{4} \right\rfloor - \left\lfloor \frac{y}{100} \right\rfloor + \left\lfloor \frac{y}{400} \right\rfloor + 2, \end{aligned}$$

where

d = the day of the month of the given date;
m = the number of the month of the year,



illustratie: Brad Haak

with January and February regarded as the 13th and 14th months of the previous year. The other months are numbered 3 to 12 as usual; and

y = the year.

This formula tells you on which day of the week a particular date in the Gregorian calendar will occur, even years into the future. Here the $\lfloor\ \rfloor$ symbol means to round the number inside it down to the nearest integer. For instance, today is 7 Feb 2004, so $d = 7$, $m = 14$, and $y = 2003$, and so

$$\begin{aligned} W =& 7 + 2(14) + \left\lfloor \frac{3(14+1)}{5} \right\rfloor + 2003 + \\ &+ \left\lfloor \frac{2003}{4} \right\rfloor - \left\lfloor \frac{2003}{100} \right\rfloor + \left\lfloor \frac{2003}{400} \right\rfloor + 2, \\ =& 7 + 28 + 9 + 2003 + 500 - 20 + 5 + 2 \\ =& 2534 \end{aligned}$$

Once W is computed, divide by W by 7 and the remainder is the day of the week with Saturday $= 0$, Sunday $= 1$, and so on, until Friday $= 6$. That is, determine the least nonnegative residue of W modulo 7. We find that $2534 = 7 \cdot 362$, so the remainder is 0 and today is Saturday.

Algebra is the language of relationships among quantities

Everyone should know the meanings of certain mathematical relationships that are more and more present in everyday language:

growing exponentially
growing logarithmically
varying directly
varying inversely
inverse square law
line of best fit (statistics, but also algebra)
rate at which a rate is changing (calculus, but also algebra)
approaching asymptotically
extrapolation, interpolation

Algebra is a language for solving certain types of problems

It used to be that the only problems that involved translation from words into mathematics in algebra texts — the so-called 'word problems' or 'story problems' involved age, motion, coins, work, and mixtures. Now textbooks are just as likely to have problems that involve everyday situations. How much of a particular food can you eat and stay within a particular diet? If the population of a town is x and growing at some rate, what will the population be some years from now?

When there is a formula relating two quantities, if you know one quantity, you can find the other, and today's technology has made it possible to work with far more complicated formulas than one could deal with even 25 years ago. The use of a graphing calculator makes it possible to find solutions to practical problems through graphing even when we do not have an algorithm to solve the equation algebraically. Graphs of functions and other relationships among variables are the geometry of algebra, and this geometry would not exist without the algebra that drives it.

Algebra is the study of structures with certain properties

An algebraic formula such as $A = LW$ would not have much advantage over its translation into words (area = length times width) were it not for the fact that operations on numbers have properties which enable formulas and other equations to be manipulated. As we all know, when $A = LW$, then (by dividing both sides both sides by L), $\frac{A}{L} = W$. What this means is that from one formula we can deduce others. So, because we know algebra, we do not have to memorize a separate formula to determine a side of a rectangle from its area. The power of deduction is not appreciated much by the uneducated public, but it is an extraordinarily powerful tool. The power of deduction is related to the next reason for algebra.

Algebra shows that our universe possesses order

We are all familiar with the use of algebra to explain 'number tricks', such as telling whether an integer is divisible by 9 by adding its digits. But algebra does more than just explain why number tricks work; its language explains many aspects of our universe. It is no coincidence that within 100 years after Viète's work in algebra, Newton and Leibniz independently developed the calculus, Newton could use this calculus to explain Kepler's laws, and so we learned why the orbits of the planets are essentially elliptical. Algebra helps to explain what to expect from the random flipping of coins, the odds of winning a lottery, whether a building can withstand the weights and other forces that will act on it, how to ship oil around the world with the least cost, how long it will take for the Earth's population to double at various rates of growth, and myriads of other activities occurring every day.

All of the prior examples involve everyday real events. If we are asking about algebra for *all* students, our case is not strengthened by appealing to physics or other subjects that students who do not take algebra would also probably not take. Even so, the following reason for studying algebra is of fundamental importance.

Algebra is a prerequisite for virtually all other mathematics

In a book simply titled *Why Math?*, Rodney Driver [4] covers a wide range of areas of mathematics, including arithmetic, algebra, geometry, vectors, combinatorics, and probability. But common to all these areas is algebra. The soul of mathematics may lie in geometry, but algebra is its heart.

Thus, without a knowledge of algebra, a person

- loses control over parts of his or her life,
- must rely on other people to do these things for him or her,
- is more likely to make unwise decisions, and
- will not be able to understand many everyday ideas, and
- will not be able to comprehend ideas dis-

cussed in chemistry, physics, the earth sciences, economics, business, psychology, and many other areas.

Thus, if two of the purposes of schooling are to create informed citizens and to optimize the work opportunities for our students, then the study of algebra is a must.

Why can so many live without it?

If algebra is so important, why have so many adults — even educated adults — been able to live nicely without it?

It is common for adults today to speak of algebra and other mathematics beyond arithmetic as if they are important only to a few people. For instance, here is a quote from a Chicago daily newspaper a few years ago:

"When the regular season begins in three weeks, Friday night's Bulls preseason opener will become about as significant as algebra formulas learned in high school." [5]

What was John Jackson, the sportswriter who wrote the article, thinking when he wrote this? To himself he must have thought, "The preseason opener is meaningless. What is the most meaningless thing I can think of? Aha! It's algebra formulas!"

Clearly Mr. Jackson knows that many of the sports statistics that are printed in his newspaper are calculated using algebraic formulas, for he is undoubtedly a college graduate. However, these writers and other adults avoid the formulas whenever they can. They are like people who go to a foreign country but do not know enough of the language to converse with native speakers in that country. Like me in the Netherlands! Even though I do not know Dutch, I can get along, but I will never appreciate the richness of the culture, and I will not be able to learn as much as I could if I knew Dutch. In my travel here, I am forced to depend on signs that have been translated into English. At this conference (de Nationale Wiskunde Dagen 2004), I found it easier to attend the sessions that were in English and missed a lot because of that.

So it is with most adults and algebra. People can live without algebra, but as a result they cannot appreciate as much of what is going on around them. They cannot participate fully. They are more likely to make unwise decisions and will find themselves with less control over their lives. They live in the same world, but they do not see or understand as much of its beauty, structure, and mystery.

The disjuncture between algebra and school algebra

Algebra in many classrooms does not at all present the picture of the vibrant, widely applicable subject that I have described. Instead of being taught as a living language with a logical structure and many connections between its topics and other subjects, algebra is taught as a dead language with a myriad of rules that seem to come from nowhere, and with applications that are viewed as puzzles, like chess problems. That so many well-educated adults wonder why they studied algebra is testimony to the disjuncture between such an important subject and the way it is presented in schools.

In the past, when only a very small percent of the population needed to learn algebra, we could be content with the algebra course as a gatekeeper. But today we cannot afford to weed out so many students. Algebra needs to *turn on* students rather than turn them off.

To remove this disjuncture, there has been, in some places a paradigm shift in the teaching and learning of algebra. This paradigm shift in attitude towards school algebra is the reason that one sees such a different approach to the subject in some contemporary materials for secondary schools. Using graphs, technology, applications, and mathematical structure, the best contemporary materials more accurately picture the *what* and the *why* of algebra than many traditional materials did. Applications are used to motivate the subject. Transitions are carefully made from arithmetic to algebra, and connections are given to geometry, statistics, and other mathematics. Algebra is changed from a skill-dominated experience, in which problems exist to practice skills, to a problem-centered experience in which skills are developed in order to solve interesting problems.

Is algebra for all theoretically possible?

It is commonly thought that mathematics has levels of abstraction. Algebra is more abstract than arithmetic. Calculus is more abstract than algebra. Higher algebra is more abstract than calculus. In a theoretical sense, in the sense of generalization, this is true. Each subject generalizes some of what has come before it. But how abstract is algebra, really. All of my work with UCSMP has convinced me that algebra is inherently no more abstract than everyday written language, but it is made more abstract by us. Learning algebra should be no more difficult than learning a new language. Any student who can learn to read possesses the ability to learn algebra.

Why do I feel this way? Because we have learned many things about algebra. First, algebra starts earlier than its formal study, whether we want it to or not. The equation $3 + \_\_ = 7$ can be considered as algebra; the use of an underline is no different than the use of a letter. So 1st or 2nd grade students do algebra problems, we just don't tell them — perhaps because we don't want to scare the teachers.

We have found no age cut-off with respect to the learning of variable. Even very young students can evaluate formulas and can graph. We should not have been surprised. Variable is supposed to be an abstract concept, but variables are introduced quite early in some countries' curricula and seem to be easier to learn early. Is the use of a letter such as A for area or x for an unknown any more abstract than the use of the letter symbol p for the sound "puh"? Surely there is an age cutoff; babies will not learn variables. But at the secondary level, from grade 6 or 7 up, it seems that the earlier the better.

Second, the use of applications concretizes algebra, motivates it, makes it easier. We know that algebra can be approached theoretically, such as through field properties, but this approach does not work with many students. On the other hand, we also know that we can approach algebra through formulas and through generalizations of patterns, and that this approach does work. It isn't automatic; competence does not come in one day or even one week or one month. But situating algebra in contexts which give reasons for studying the subject at the same time that they illustrate the concepts changes a person's view of algebra forever. I know that many of you here could not return to the way you used to teach algebra.

But let me raise a caution. I do not wish to be interpreted as suggesting that you can just go into a classroom of students and teach them algebra in some 'correct way' and they will learn. There are prerequisites to learning algebra. If a symbol is to stand for a number, as variables usually do, you have to know something about number. You have to know that a number can be represented in many ways, that 6 can be written as $\sqrt{36}$ and as $12/2$ and as 6.000, and that if $x = 6$, then x can be written in any of those ways, too. You need to know what it means for one number to be close to another in value. If you have the expression $x + y$, you need to know what the + sign means independently of what the numbers x and y are. These and other arithmetic ideas are necessary for success in algebra. But they can be learned by virtually all students.

For these reasons, I believe that virtually all students can learn algebra. But is it realistically possible?

Achieving algebra for all

To achieve algebra for all students, I think that we have to change the attitudes toward algebra that most of the public have. To do this, it helps to look at how arithmetic became a subject that everyone felt they needed to learn.

Remember that arithmetic, too, is a relatively new subject. The algorithms that we use for multiplication and division were invented mostly in Italy in the 1400s. Expecting everyone to be competent in the four fundamental operations is a phenomenon of the past 300 years. It took great numbers of people participating in trade and using numbers everyday, the invention of numeration systems and algorithms — e.g., the use of decimals by Simon Stevin – that made it relatively easy for people to compute, and the widespread availability of paper and pencil (or pen and ink) that made it possible to record the work. Do we have that for algebra? Can we have that for algebra?

The answer is that we are closer than most people think. A large percentage of the people who use a computer today work with spreadsheets. They do not think they are doing algebra when they use *Excel* and other spreadsheets because this does not look like school algebra. To most people, algebra is supposed to be hard. Algebra is supposed to be something you do not understand.

But working with a spreadsheet *is* algebra. The names of the variables are $AI, A2, A3, \ldots, B1, B2, \ldots$ Each time we ask the spreadsheet to calculate something and put it in another cell we are writing an algebraic formula. For example, if in cell $C1$ we type "$= A1 + B1$", then we are naming the variable $C1$. The spreadsheet does the calculation for us, of course. Each time that we copy a formula to other cells, we are creating a function. The function may be explicitly defined or it may be an iterative sequence, but it is a function either way. Most spreadsheet programs allow us to graph the explicitly defined functions. With either the graph or by successive approximation, we can solve many equations. Also, almost all spreadsheet programs have a form of summation notation that makes for a very easy transition to traditional $\sum$-notation. Sum(A1:A5) is so similar to $\sum_{i=1}^{5} A_i$ Thus algebra capability is in most every computer, on virtually every business desk. People who think they do not know algebra are using this capability.

Spreadsheets do not do all that we view as algebra. But most that is left can be done by computer algebra systems (CAS). Want to factor a trinomial? Then put the trinomial into a TI-92 or Casio-cfx 9700g and give the instruction to factor. Want to solve a cubic exactly? Put the cubic into the CAS system and give the instruction to solve. Want to differentiate a function? Want to find a definite or indefinite integral? Want to solve a differential equation? For more complex mathematics, you may need *Mathematica* or *Maple*, but if there is an algorithm for the process, then there is a CAS system that performs that algorithm.

Paper-and-pen and, more recently, paper-and-pencil has been the technology that we have used for the past 400 years to do arithmetic. Paper-and-pen became dominant not because people understood why the algorithms work. They still don't! Paper-and-pencil won because its algorithms were more widely applicable. I believe we are now in the same situation with computers and algebra. In time, the computer algorithms embodied in spreadsheets and automatic graphers and CAS systems will become the algebra that everyone uses and recognizes. Spreadsheets will be employed to help introduce the language of algebra to students, and CAS systems will be used to perform the complex manipulations.

This does not mean that paper-and-pencil algebra will become obsolete. They will not become obsolete any more than mental arithmetic has become obsolete. People will still need to know how to solve simple equations and do simple manipulations by hand. And, perhaps more important, they will need to be able to translate from real and fanciful situations to mathematics and vice-versa. Algebra is more important than ever. School algebra is not obsolete, but it needs to change to touch the everyday lives of students and their families in order to be relevant to all students, and it needs to recognize its relationships with technology in order for the public to be comfortable with the notion of algebra for all.

Many if not most mathematics teachers in the world do not have much experience with this paradigm change in algebra. Even some talented mathematicians and many of the best mathematics teachers in the United States and those most involved with technology do not view algebra in this light. I know that many, most, and perhaps all of you are very much into the use of applications and technology in the teaching of mathematics, and I hope that my remarks will be of use to you in your teaching and in talking to others who may not be as knowledgeable or as fond of mathematics as we are.

Referenties

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