Problemen/UWC

Universitaire Wiskunde Competitie

De Universitaire Wiskunde Competitie (UWC) is een ladderwedstrijd voor studenten, georganiseerd in samenwerking met de Vlaamse Wiskunde Olympiade. De opgaven worden tevens gepubliceerd op de internetpagina http://academics.its.tudelft.nl/uwc

Ieder nummer bevat de ladderopgaven A, B, en C waarvoor respectievelijk 30, 40 en 50 punten kunnen worden behaald. Daarnaast zijn er respectievelijk 6, 8 en 10 extra punten te winnen voor elegantie en generalisatie. Er worden drie editieprijzen toegekend, van 100, 50, en 25 euro. De puntentotalen van winnaars tellen voor 0, 50, en 75 procent mee in de laddercompetitie. De aanvoerder van de ladder ontvangt een prijs van 100 euro en begint daarna weer onderaan. Daarnaast wordt twee maal per jaar een ster-opgave aangeboden waarvan de redactie geen oplossing bekend is. Voor de eerst ontvangen correcte oplossing van deze ster-opgave wordt eveneens 100 euro toegekend.
Groepsinzendingen zijn toegestaan. Elektronische inzending in LaTeX wordt op prijs gesteld. De inzendtermijn voor de oplossingen sluit op 1 mei 2004. Voor een ster-opgave geldt een inzendtermijn van een jaar.
De Universitaire Wiskunde Competitie wordt gesponsord door Optiver Derivatives Trading en wordt tevens ondersteund door bijdragen van de Nederlandse Onderwijs Commissie voor Wiskunde en de Vereniging voor Studie- en Studentenbelangen te Delft.

Opgave A

For every integer $n > 2$ prove that

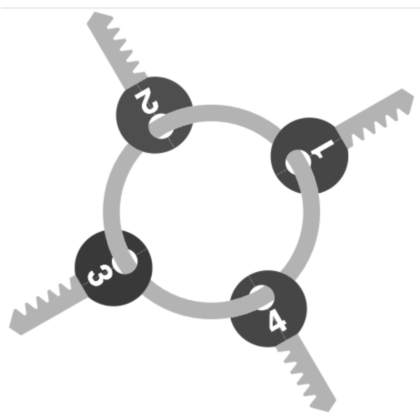
$$\sum_{j=1}^{n-1}\left(\frac{1}{n-j}\sum_{k=j}^{n-1} 1/k\right) < \pi^2/6.$$

Opgave B

Consider the first digits of the numbers 2^n: $1, 2, 4, 8, 1, 3, 6, 1, 2, 5, 1, 2, 4, \ldots$. Does the digit 7 appear in this sequence? Which digit appears more often, 7 or 8? How many times more often?

Opgave C

We have a circular key chain and we want to colour the keys, using as few colours as possible, so that each key can be identified by the colour pattern — that is, by looking at the key's colour and neighboring colours as far away as needed. Let $f(n)$ be the minimal number of colours required to uniquely disambiguate a circular key chain of n keys in this way. Determine $f(n)$ for all positive integers n.



Eindredactie: Robbert Fokkink en Jan van Neerven
Redactieadres: Universitaire Wiskunde Competitie
Faculteit Electrotechniek
Technische Universiteit Delft
Postbus 5031, 2600 GA Delft
J.vanNeerven@math.tudelft.nl

Oplossingen

Ster-opgave

For $n \in \mathbf{N}\backslash\{0\}$ and $x \in \mathbf{R}$ define

$$P_n(x) := n^n x((x+1)^{n+1}-1)^{n-1} - (n+1)^{n-1}((x+1)^n-1)^n.$$

For $n \geq 2$, is it true that this polynomial is of the form

$$P_n(x) = \sum_{k=n+2}^{n^2} c_{n,k} x^k$$

with $c_{n,k} > 0$ for $n+2 \leq k \leq n^2$?

Editie 2003/3

Op de ronde 2003/3 van de Universitaire Wiskunde Competitie ontvingen we inzendingen van Filip Cools, Kenny De Commer, Syb Botma, Sven Vanhoecke, Hendrik Hubrechts en Roelof Oosterhuis.

Opgave 2003/3-A

Let $(a_n)_{n=0}^{\infty}$ be a non-decreasing sequence of real numbers such that $(n-1)a_n = na_{n-2}$ for $n = 2, 3, \ldots$ with initial value $a_0 = 2$. Compute a_1.

Oplossing Several people let us know that they enjoyed this problem, which is due to Alexandre Lupasz. We received hors concours solutions from Klaas Pieter Hart, Alex Heinis, Edward van Kervel, Ton Kool, Jack van Lint, Ludolf Meester, Jaap Spies and Vincent de Valk. Some use the gamma function and others use Wallis's product. Here is Edward van Kervel's solution. With induction, using the fact that the a_n are non-decreasing, we obtain

$$2 \cdot \frac{2}{1} \cdot \frac{4}{3} \cdots \frac{2n}{2n-1} \leq a_1 \cdot \frac{3}{2} \cdot \frac{5}{4} \cdots \frac{2n+1}{2n} \leq 2 \cdot \frac{2}{1} \cdot \frac{4}{3} \cdots \frac{2n}{2n-1} \cdot \frac{2n+2}{2n+1}$$

which can be rewritten as

$$2 \cdot \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdots \frac{2n}{2n-1} \cdot \frac{2n+1}{2n} \leq a_1 \leq 2 \cdot \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdots \frac{2n}{2n-1} \cdot \frac{2n}{2n+1} \frac{2n+2}{2n+1}.$$

The left-hand product and the right-hand side converge to the same infinite product as $n \to \infty$. This is Wallis's product, which is equal to π.

Opgave 2003/3-B

Let $S(n)$ be the sum of the remainders on division of the natural number n by $2, 3, \ldots, n-1$. Show that

$$\lim_{n \to \infty} \frac{S(n)}{n^2}$$

exists and compute its value.

Oplossing Klaas Pieter Hart has sent us a self contained solution, which is similar to the solution by Roelof Oosterhuis that is given below. Jaap Spies uses a shortcut by invoking a result from Hardy and Wright. Let

$$S(n) = \sum_{i=2}^{n} (n \bmod i) = \sum_{i=2}^{n} n - i \left\lfloor \frac{n}{i} \right\rfloor .$$

Notice that $0 \leq (n \bmod i) \leq i-1$. We write $S(n) = T(n) + U(n)$, where

Oplossingen

$$T(n) = \sum_{i=2}^{\lfloor\sqrt{n}\rfloor} (n \bmod i), \quad U(n) = \sum_{i=\lfloor\sqrt{n}\rfloor+1}^{n} (n \bmod i)\,.$$

By our choice of $T(n)$ we can estimate $T(n)$ by

$$0 \le T(n) \le \sum_{i=2}^{\lfloor\sqrt{n}\rfloor} (i-1) = \frac{1}{2}(\lfloor\sqrt{n}\rfloor - 1)\lfloor\sqrt{n}\rfloor \le \frac{1}{2}n$$

which implies that

$$\lim_{n\to\infty} \frac{T(n)}{n^2} = 0\,.$$

In order to estimate $U(n)$, we denote by $\alpha_k = \lfloor \frac{n}{k} \rfloor$. For $\alpha_{k+1} < i \le \alpha_k$ we have $\lfloor \frac{n}{i} \rfloor = k$. Let

$$L_k = \sum_{i=\alpha_{k+1}+1}^{\alpha_k} n \bmod i\,.$$

We can write $U(n)$ as a sum of L_k by

$$U(n) = \sum_{k=1}^{\lfloor\sqrt{n}\rfloor} L_k\,.$$

We calculate

$$L_k = \sum_{i=\alpha_{k+1}+1}^{\alpha_k} n - ik = (\alpha_k - \alpha_{k+1})\left(n - \frac{k}{2}(\alpha_k + \alpha_{k+1} + 1)\right)\,.$$

Now we can estimate L_k by

$$\left(\frac{n}{k(k+1)} - 1\right)\left(\frac{n}{2(k+1)} - \frac{k}{2}\right) < L_k < \left(\frac{n}{k(k+1)} + 1\right)\left(\frac{n}{2(k+1)} + \frac{k}{2}\right).$$

Notice that we are able to bound $\frac{L_k}{n^2}$ by

$$\left|\frac{L_k}{n^2} - \frac{1}{2k(k+1)^2}\right| < \frac{1}{n(k+1)} + \frac{k}{2n^2}\,.$$

Since

$$\sum_{k=1}^{r-1} \frac{1}{n(k+1)} + \frac{k}{2n^2} < \frac{\log(\sqrt{n})}{n} + \frac{\frac{1}{2}(\sqrt{n}-1)\sqrt{n}}{n^2}\,,$$

which will vanish as $n \to \infty$, we find that the limit for $\frac{L_k}{n^2}$ exists.
Finally we conclude that

$$\begin{aligned}\lim_{n\to\infty} \frac{S(n)}{n^2} &= \lim_{n\to\infty} \frac{U(n)}{n^2} = \lim_{n\to\infty} \sum_{k=1}^{\lfloor\sqrt{n}\rfloor} \frac{L_k}{n^2} = \lim_{n\to\infty} \sum_{k=1}^{\lfloor\sqrt{n}\rfloor} \frac{1}{2k(k+1)^2} \\ &= \frac{1}{2}\sum_{k=1}^{\infty}\left(\left(\frac{1}{k} - \frac{1}{k+1}\right) - \frac{1}{(k+1)^2}\right) \\ &= \frac{1}{2}\left(1 - \sum_{k=2}^{\infty}\frac{1}{k^2}\right) = \frac{1}{2}\left(1 - (\frac{\pi^2}{6} - 1)\right) = 1 - \frac{\pi^2}{12}\end{aligned}\,.$$

Opgave 2003/3-C
Consider four distinct points A, B, C, D in the plane such that the line segments AB, BC, CD, DA do not intersect. Let Z_1, Z_2, Z_3 be the centers of gravity of:
1. the four point masses in A, B, C, D.
2. the wire frame with vertices A, B, C, D.
3. the plate with corners A, B, C, D.

If A, B, C, D form a parallelogram then Z_1, Z_2, Z_3 coincide. Does the converse hold?

Oplossing This problem was proposed by Karel Post. It has been solved by Jaap Spies and by Roelof Oosterhuis. The elegant solution of Oosterhuis is as follows.

Define a coordinate system whose origin is at the middle of A and C, and let $A, B, C, D \in \mathbf{R}^2$. We may assume that the line segment AC lies inside the quadrangle $ABCD$ (if this is initially not the case we can rename the points).
Assume that $Z_1 = Z_3$. We have to prove that $ABCD$ is a parallelogram, or equivalently, that $B + D = A + C$ which is, by definition, zero.
We have

$$Z_1 = \frac{B + D}{4}$$

The center of mass of the plate lies between the centers of mass of the triangles ABC and ACD (respectively $B/3$ and $D/3$) and its location depends linearly on their areas. Hence,

$$Z_3 = \frac{1}{3}\frac{Br + Ds}{r + s}$$

where r denotes the distance between B and the line through AC, and s the distance between D and that line. We know $Z_3 - Z_1 = 0$ and therefore

$$(r - 3s)B + (-3r + s)D = 0 \tag{1}$$

which means that B and D are linearly dependent (the coefficients can not be both zero for nonzero areas), hence $D = \lambda B$ for some $\lambda < 0$ (since AC lies inside $ABCD$) and therefore $s = -\lambda r$. Substituting this into (1) we obtain

$$(1 - \lambda^2)rB = (1 - \lambda)r(B + D) = 0$$

which proves that $B + D = 0 = A + C$.
We did not need the center of gravity of the frame!

Uitslag Editie 2003/2
De weging van de opgaven is 3 : 4 : 5.

	Naam	A	B	C	*Totaal*
1.	Roelof Oosterhuis (Groningen)	10	8	10	112
2.	Kenny De Commer (Leuven)	10	8	0	62
3.	Syb Botsma (Utrecht)	8	7	-	52
4.	Hendrik Hubrechts (Leuven)	10	5	-	50
	Filip Cools (Leuven)	10	5	-	50
5.	Sven Vanhoecke (Brussel)	6	0	-	18

Ladderstand Universitaire Wiskunde Competitie
We vermelden alleen de top 3. Voor de complete ladderstand verwijzen we naar de UWC-website.

	Naam	*Punten*
1.	Filip Cools	249
2.	Syb Botsma	140
3.	Tom Claeys	138