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Boekbespreking The collected works of P.A.M. Dirac, 1924–1948

On the verge of new mathematics

Het is honderd jaar geleden dat Paul Dirac werd geboren. Hij was een van de meest begaafde theoretische fysici die de twintigste eeuw gekend heeft. Dirac gaf een geheel relativistische beschrijving van het electron en voorspelde het bestaan van een positron. In 1933 wonnen Schrödinger en hij de Nobelprijs voor de fysica. Ook heeft hij wezenlijke bijdragen geleverd aan diverse gebieden binnen de wiskunde zoals de distributietheorie, spectraaltheorie en de theorie van partiële differentiaalvergelijkingen. Het eerste deel van zijn verzamelde werken is in 1995 verschenen en wordt besproken door Leo van Hemmen, hoogleraar in de theoretische (bio)fysica aan de Technische Universiteit München.

Paul Adrien Maurice Dirac was one of the most prominent theoretical physicists of the 20th century and had at the same time a deep and lasting influence on mathematics. He was born in Bristol on August 8, 1902. In 1918 he entered the University of Bristol and became a student of electrical engineering. A year later his father and the children gave up their Swiss nationality and became British citizens by naturalization. He graduated as an electrical engineer at Bristol in 1921 and two years later in mathematics, both times with 1st class honors. In October 1923 he became a postgraduate student at St. John's College, Cambridge, and began research in theoretical physics under the supervision of the widely respected R.H. Fowler at the Cavendish Laboratory.

Commutation relations

In May 1926 Dirac submitted a dissertation entitled *Quantum Mechanics* after Fowler in August 1925 had shown him galley proofs of Heisenberg's breakthrough in quantum mechanics [1]; see van der Waerden [2] for a translation of Heisenberg's paper and an excellent short account of the historical developments. December 1st of the same year Dirac published his first paper on the subject. Here he introduced the notion of 'commutator' $[q, p] = qp - pq$ for two operators q and p defined, as we would now say, on a suitable domain. The commutator has been a mainstay of quantum mechanics ever since. Furthermore, he coined the problem of how to represent commutation relations [3] such as $[q_i, p_j] = i\hbar\delta_{ij}$ where the q_i and p_j are finitely many self-adjoint operators with the p's and q's commuting among each other, δ_{ij} is the Kronecker delta, and $\hbar > 0$ is Planck's constant h divided by 2π. There is a simple solution in the one-dimensional case where $q := x$ is a coordinate, $p := (\hbar/i)\mathrm{d}/\mathrm{d}x$ the momentum (operator) belonging to it, and, evidently, $[q, p] = i\hbar$. In more than one dimension, its analogue directly gives an explicit representation of the commutation relations $[q_i, p_j] = i\hbar\delta_{ij}$. One may then ask: what is it good for and is it the only one?

It is good for quantum mechanics, but what is quantum mechanics? Let us quote Dirac himself [4]: "We now make the further assumption that *linear operators correspond to the dynamical variables at a cer-*

tain time. By dynamical variables are meant quantities such as the coordinates and the components of velocity, momentum and angular momentum of particles, and functions of these quantities — in fact the variables in terms of which classical mechanics is built up. The new assumption requires that these quantities shall occur also in quantum mechanics, but with the striking difference that *they are now subject to an algebra in which the commutative axiom of multiplication does not hold*." Instead the algebra is determined by prescribing commutators, for example, as indicated above. The next step consists of introducing a dynamics based on the classical Hamiltonian; an example will be given below. Once the dynamical variables and their dynamics have been specified, the dynamical system as such is fully determined. Until now quantum mechanics has successfully explained all phenomena at the atomic level.

Dirac simply posed the representation problem and indicated a solution. He never bothered about uniqueness and other intrinsically mathematical questions. It was von Neumann who showed as early as 1931 that *finitely* many $U_i(s) = \exp(isq_i)$, $V_j(t) = \exp(itp_j)$, satisfying $U_i(s)V_j(t) = \exp(i\hbar st\delta_{ij})V_j(t)U_i(s)$, are essentially unique in that they are the ones given above or a direct sum thereof. (Years later Rellich and Dixmier brought p and q back to the 'floor' where they came from [3].)

The idea behind the problem of finding a commutator representation is that quantum-mechanical commutators ought to be derivable from their classical counterparts, the Poisson brackets, by multiplying the latter by $i\hbar$; this rule proposed by Dirac is physically deep — and very elegant. Finally, in the very same 1926 paper referred to above, he indicated how time-dependent perturbation theory for a self-adjoint operator and Fermi-Dirac statistics are to be treated.

The Dirac equation

In 1928 he introduced the 'Dirac equation', the first relativistic formulation of the quantum mechanics of the electron. The paper introducing it makes for fascinating reading and utmost elegance [5]. Dirac eliminated his equation's drawback of having negative energies without lower bound by a solution that led to the 'anti-electron', i.e., the positron, which should have the same mass as the electron but opposite charge; it was discovered by Anderson in 1933. The year before he was elected Fellow of the Royal Society — a bit late, if one realizes that our present formulation of quantum mechanics and Dirac's unification of two different approaches, namely, the matrix mechanics due to Born, Heisenberg and Jordan and the Schrödinger equation with its wave function, was completely finished by then. Dirac and Schrödinger were jointly awarded the Nobel Prize for physics in 1933. A year earlier, in 1932, Dirac had been elected Lucasian Professor of Mathematics at the University of Cambridge. One of his predecessors in the chair was Newton. He held the position until his retirement in 1969, when he moved to Tallahassee, Florida. Here he died on October 20, 1984. He was also buried there. He was survived by his wife Margit Wigner, whom he had married in 1937; she was a sister of Eugene P. Wigner, a master of group theory [6] and another physics Nobel Prize winner. There is the rumor that, when a friend was visiting them shortly after their marriage, Dirac introduced his wife by saying "This is Wigner's sister."

Distribution theory

The influence of Dirac on the development of both physics and mathematics is immense. In addition to his research papers, his book *The principles of quantum mechanics* [4] had an impact that is hard to overestimate; its second edition published in 1935 was the most influential. I now list a few of his main ideas. The *Dirac delta function* δ, which is not a function but a distribution, was the key theme in the prelude to the theory of distributions developed by Laurent Schwartz in the late forties [8]. Dirac defined his delta function $\delta(x-a)$ to have support $\{a\}$ and, for any continuous function f, satisfy the equality

$$\int_{-\infty}^{\infty} \mathrm{d}x\, f(x)\delta(x-a) = f(a)\,. \tag{1}$$

Furthermore, he noticed (see [4]) that it 'appears whenever one differentiates a discontinuous function.' One might argue that Heaviside also played a role but, in fact, it is a negligible one since he did his, no doubt innovative, work half a century earlier (1893–94) without anybody really worrying about its mathematical meaning [7]. Dirac's masterful usage of the delta function changed the scene completely and established the need for a rigorous justification.

Spectral theory

The delta function also played a dominant role in Dirac's spectral theory of 'generalized' eigenfunctions. Let us consider, for instance, the coordinate $q := x$ and the momentum $p := (\hbar/i)\,\mathrm{d}/\mathrm{d}x$ on the real line, two self-adjoint operators obeying $[q,p] = i\hbar$ on a suitable dense domain [3, 10] in $L^2(\mathbf{R})$. It is plain that q has been given its spectral representation where it is 'diagonal'. Its counterpart p, however, is not but its Fourier transform is. To wit, Dirac observed that plane waves $\exp(ikx)$ are its eigenfunctions, $p\exp(ikx) = \hbar k \exp(ikx)$. Mathematically, they are not since the Hilbert space $L^2(\mathbf{R})$ is too small to contain them. They are *generalized* eigenfunctions, a notion that can be made precise in terms of a Gelfand triple [9] consisting of a smaller space V in the self-dual Hilbert space $\mathcal{H}$ contained in a larger dual space $V^\star : V \subset \mathcal{H} = \mathcal{H}^\star \subset V^\star$. It is the space $V^\star$ that contains generalized eigenfunctions such as $\exp(ikx)$. The argument that now comes is a specimen of Dirac's mathematical elegance. The original idea was published as a two-page argument just before the second world war.

A complex Hilbert space $\mathcal{H}$ is a linear vector space that is complete with respect to the norm induced by the inner product $\langle\cdot|\cdot\rangle$. It is taken to be linear in the *right*-hand side. We will soon see that there is a good reason for doing so. Dirac called $\langle$ a 'bra' and $\rangle$ a 'ket' since the inner product $\langle\cdot|\cdot\rangle$ looks like a bracket. A good notation leads half the distance to a good result. Let then $\mathcal{H}$ be finite-dimensional and let Λ be a self-adjoint operator in $\mathcal{H}$. Its eigenvalues are called λ, the corresponding eigenvectors $|\lambda\rangle$ satisfy $\Lambda|\lambda\rangle = \lambda|\lambda\rangle$, and they are normalized so that $\langle\lambda|\lambda'\rangle = \delta_{\lambda\lambda'}$. The key question Dirac wanted to answer was: what does the spectral representation of Λ look like? Using his bracket notation we readily see

$$\Lambda = \textstyle\sum_\lambda |\lambda\rangle\,\lambda\,\langle\lambda| \tag{2}$$

where the sum is over all eigenvalues λ. The corresponding decomposition of unity, which is running under the name 'completeness of the eigenfunctions', is

$$\mathbf{1} = \textstyle\sum_\lambda |\lambda\rangle\langle\lambda|\,. \tag{3}$$

The proof of the pudding consists in noting $(|\lambda\rangle\langle\lambda|)|x\rangle = \langle\lambda|x\rangle|\lambda\rangle$. To verify (2) it suffices to let it operate on each of the eigenvectors; the result is evidently true. Moreover, we have obtained a suggestive way of writing projection operators. For what follows we observe that $\mathbf{1}$ is a matrix with elements $\mathbf{1}_{ij} = \delta_{ij}$.

We now return to $L^2(\mathbf{R})$ and ask the same question as in (2) for the operator $p = (\hbar/i)\,\mathrm{d}/\mathrm{d}x$. Let

$$(\mathcal{F}f)(k) = \int \exp(ikx) f(x)\,\mathrm{d}x/\sqrt{2\pi}$$

be the Fourier transform with inverse $\overline{\mathcal{F}}$ so that $\overline{\mathcal{F}}\mathcal{F}$ is the identity operator. Dirac wrote

$$f(y) = (\overline{\mathcal{F}}\mathcal{F}f)(y) = \int \mathrm{d}x\, f(x) \left\{ \int \frac{\mathrm{d}k}{2\pi} \exp[ik(x-y)] \right\} \tag{4}$$

and concluded

$$\int \frac{\mathrm{d}k}{2\pi} \exp ik(x-y) = \delta(x-y) \tag{5}$$

so as to obtain the analogue of (3) — a relation that is evident in the context of distribution theory [8]. What do we gain? A lot, if, following Dirac, we consider a delta function as the 'unit' for multiplying functions in (a dense subset of) $L^1(\mathbf{R}) \cap L^2(\mathbf{R})$, which with the benefit of hindsight is taken to be a convolution algebra, interpret (5) as the analogue of (3), use Dirac's notation,

$$\mathbf{1} = \int \frac{\mathrm{d}k}{2\pi} \, |\exp(ikx)\rangle\langle\exp(iky)| \,, \tag{6}$$

write $|k\rangle := |\exp(ikx)\rangle$, and note

$$p = p\mathbf{1} = \int \frac{\mathrm{d}k}{2\pi} \, |k\rangle\, \hbar k \,\langle k| \tag{7}$$

since $p \exp(ikx) = \hbar k \exp(ikx)$. That is to say, $\exp(ikx)$ is now a *generalized* eigenfunction and (7) corresponds to (2). Mathematically [3], one could say that the Fourier transform is a diagonalizing transformation in that $\overline{\mathcal{F}} p \mathcal{F} = \hbar k$ but Eq. (7) is far more suggestive. It was Dirac's genius that wrote down a suggestive notation and derived spectral representations that were given their mathematical justification [9] years later. Since Dirac was right, why complain?

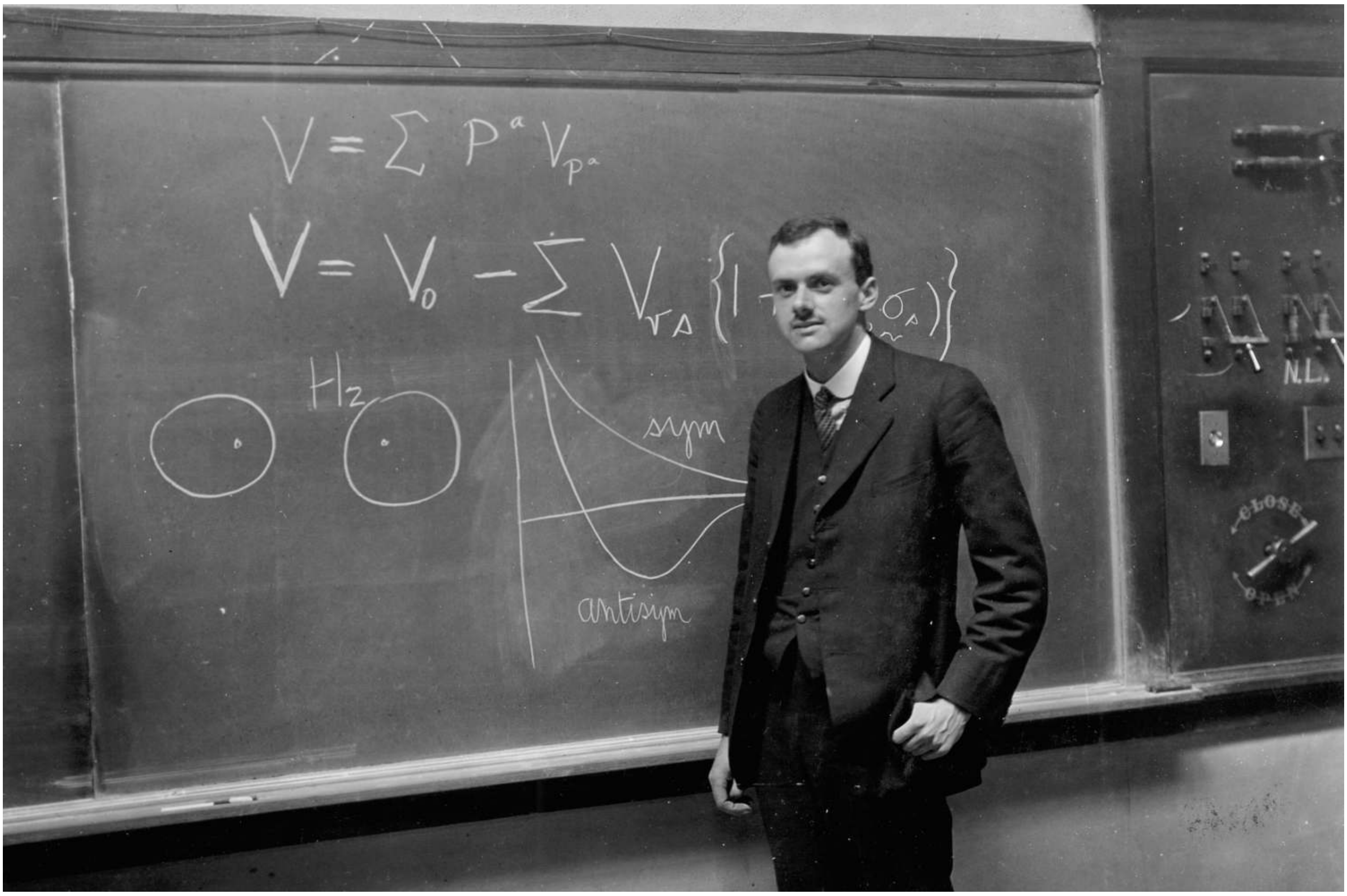
Feynman-Kac formula

Also for what is now called the Feynman-Kac formula, Dirac laid the foundations. We sketch the idea and refer to the literature [10–12] for technical details. Let us consider a particle with mass $m = 1$ in a regular external potential V defined on $\mathbf{R}^3$. The particle's Hamilton function ('energy') is $(p_x^2 + p_y^2 + p_z^2)/2 + V$. Its quantum-mechanical equivalent is the Hamilton operator $H = -\hbar^2 \triangle/2 + V$ with the negative Laplacian $-\triangle$ stemming from the substitution $p_x \mapsto (\hbar/i)\partial_x$, etc. The time evolution of the particle's wave function, belonging to $\mathcal{H} = L^2(\mathbf{R}^3)$, is given by the unitary operator $\exp(-itH/\hbar)$. The Trotter product formula [13] then tells us

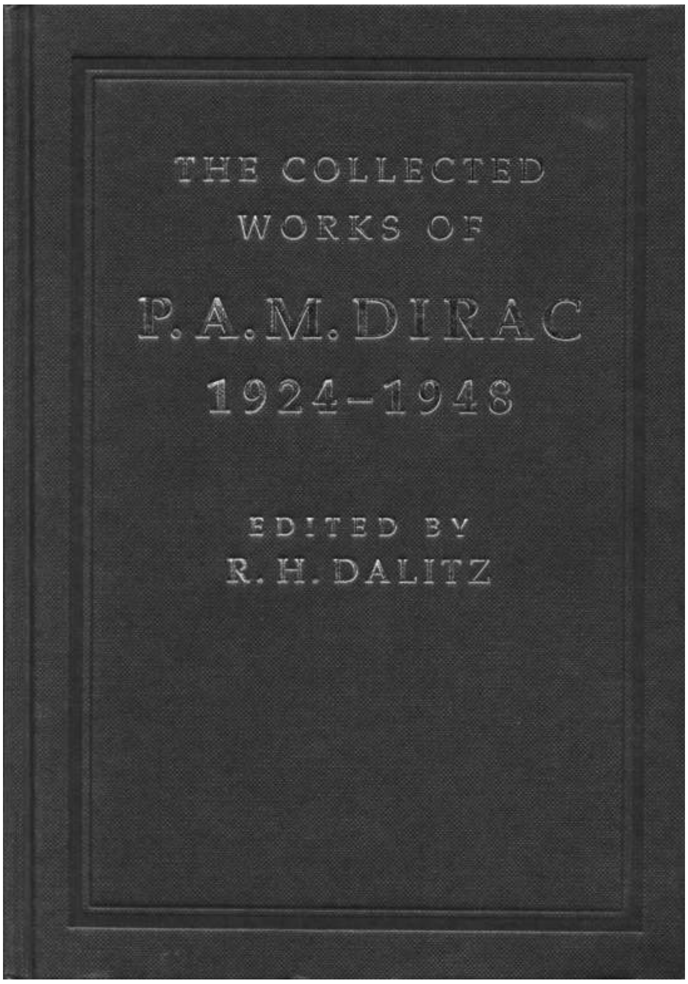
$$\exp(-itH/\hbar) = \lim_{n\to\infty} \left[e^{i(\hbar t/2n)\triangle} \, e^{-(it/n\hbar)V} \right]^n . \tag{8}$$

For the three-dimensional Laplacian there exists an explicit representation of the time evolution as an integral operator,

$$\left[e^{i(\hbar t/2n)\triangle} f \right](x) = \left(\frac{n}{2\pi i \hbar t} \right)^{3/2} \int \mathrm{d}y \, \exp\left(\frac{in\|x-y\|^2}{2\hbar t} \right) f(y) \,. \tag{9}$$



Paul Dirac in the thirties, explaining 'exchange interaction' in the quantum mechanics of the hydrogen molecule H_2 as a consequence of its two electrons being *identical* particles obeying Fermi-Dirac statistics; cf. §58 of the 4th edition of his classic [4] on quantum mechanics. The equation on the blackboard is to be Eq. (32) of the book, viz., $V = V_0 - (1/2)\sum_{r<s} V_{rs}\{1 + (\sigma_r, \sigma_s)\}$ with σ representing either particle's spin, a vector, and (σ_r, σ_s) standing for their inner product. As it behooves a good theoretician, the prefactor $1/2$ is missing on the blackboard.



Combining (8) and (9) we find

$$[\exp(-itH/\hbar)f](x_0) = \lim_{n\to\infty} \left\{ \left(\frac{n}{2\pi i\hbar t}\right)^{3n/2} \int \dots \int \mathrm{d}x_1 \dots \mathrm{d}x_n \right\} \exp[iS_n(x_0, x_1, \dots, x_n; t)/\hbar] f(x_n) \tag{10}$$

where the x_k are taken at times $t_k = kt/n$ and

$$S_n(x_0, x_1, \dots, x_n; t) := \sum_{i=1}^{n} \frac{t}{n} \left[\frac{1}{2} \left(\frac{\|x_i - x_{i-1}\|}{t/n} \right)^2 - V(x_i) \right]. \tag{11}$$

In view of the choice of the x_k and the limit $n \to \infty$, it is tempting to interpret $[\|x_i - x_{i-1}\|/(t/n)]^2$ as the velocity $v(t_i)^2$ squared, so that we are left with the *classical* kinetic energy $T = v^2/2$ and, hence, with the Lagrangian $L = T - V$ in the sum appearing in (11). As $n \to \infty$ we put $x_n := y$ and allow the sum to formally converge to the so-called action

$$S(x, y; t) = \int_0^t \mathrm{d}s\, L(x, y; s) \tag{12}$$

for a path $\{x(s);\ 0 \le s \le t\}$ starting in $x_0 := x$ and ending in y.

Returning to (10) it is now even more tempting to interpret the expression between the curly brackets as a measure, to be called ‘dω’ for the moment, write

$$[\exp(-itH/\hbar)f](x) = \int \mathrm{d}y\, K(x, y; t) f(y) \tag{13}$$

with the kernel K being given by

$$K(x, y; t) = \int \text{‘}\mathrm{d}\omega\text{’} \exp[\frac{i}{\hbar} \int_0^t \mathrm{d}s\, L(x, y, \omega; s)] \tag{14}$$

for paths ω starting at x at time $t = 0$ and ending at y at time t. This, in fact, is what Dirac wrote down as early as 1933 in a paper on *The Lagrangian in quantum mechanics* [14], long before Feynman published the results of his 1942 Ph.D. thesis in 1948 [15].

Feynman, who was aware of Dirac’s work, took the representation (14) serious and observed that in the classical limit, when quantum mechanics has to reduce to classical mechanics by taking $\hbar \to 0$, the principle of stationary phase leads to paths that are extrema of the action appearing in (14) and, hence, satisfy the Euler-Lagrange equations for this variational problem — as should be the case. In fact, though Feynman has been credited for this, Dirac [14] already said so ...

The consequences of the above circle of ideas have been very rich. Mark Kac wrote a fundamental paper [16] in 1951 showing that, if the Schrödinger equation is replaced by the diffusion equation, which simply means that it in (10) is replaced by t, then the heuristic ‘measure’ dω as postulated in (14) can be interpreted as a proper one, the Wiener measure. To be precise, the Wiener paths are continuous but *nowhere* differentiable so that the classical, time-integrated, kinetic energy $\int_0^t \mathrm{d}s\, T(x_0, x_n; s)$ makes no sense as such. Only the *combination* with the sum over paths leads to a Wiener measure μ_x over paths starting at x at time $t = 0$. What is then left from (13) is the Feynman-Kac formula

$$[\exp(-tH/\hbar)f](x) = \int \mathrm{d}\mu_x(\omega) \exp[-\frac{1}{\hbar}\int_0^t \mathrm{d}s\, V(\omega(s))] f(\omega(t))\,. \tag{15}$$

Its impact on mathematical physics as well as probability theory has been huge [11–12] and it is good to realize where it stems from: a short paper of Dirac in the *Physikalische Zeitschrift der Sovjetunion* — a consequence of his close relationship with Russian physicists, first and foremost Peter Kapitza, who was Royal Society Professor at Cambridge’s Trinity College but was not allowed by Stalin to return to England from a visit to Moscow in 1935. Also (14) is still as ‘Feynman path integral’ a focus of intense mathematical research [17] so as to make it mathematically well-defined. Once this goal would be achieved, the classical limit $\hbar \to 0$ of quantum time evolution, such as in (14), would be an interesting corollary.

Conclusion and outlook

Dirac’s ideas have been a steady source of inspiration. Not only have they led to new physics but also to novel mathematical research. A nice example is the question of whether an electron is to be considered as a point source or as a charge distribution of finite extent. His paper *Classical theory of radiating electrons* [18] was long considered to have “spectacular deficiencies” (editor Dalitz’ original statement) because of run-away solutions and the supposedly ad hoc asymptotic condition. Recent research [19] has shown that it can all be understood and makes good sense in the context of singular perturbation theory.

As for the collected works, more than half of them stemming from the ‘golden years’ 1925–1939, the printing is as it behooves Cambridge University Press. The price, however, is not. The editor has interpreted his task in a minimal way and provided hardly more than the six pages chronology of Dirac’s life. There is no comment, no evaluation, nothing — except for the tables of contents of the different editions of Dirac’s extraordinary book on quantum mechanics [4], the English versions of the prefaces to the Russian editions, a printed version of Dirac’s hand-written original of what became Chapter XI-a of the first (1932)

Russian edition, and a copy of Fowler's 1925 note to Dirac written on Heisenberg's galley proof [1]: "What do you think of this? I shall be glad to hear." In view of the results his invitation triggered, Fowler should have been more than glad. In passing I note that about 15 percent of Dirac's early papers have been published in the Proceedings of the Royal Society (London) through Fowler: the time between submission and publication was hardly ever more than a month. Who said that present-day letter journals were fast?

To compensate the defect of missing background information [20], Dirac's wonderful style often leads to unsurpassed clarity. Though he considered himself a theoretical physicist, his style is reminiscent of a mathematician's. Despite their clarity, both his mathematics and his physics would, in my opinion, have profited greatly from expert explanation and interpretation putting them into a perspective that incorporates present-day insight; Spohn's work [19] is just an indicative example.

In summary, I have touched upon a few of the main contributions of Dirac to mathematics: distributions, spectral theory, and the Feynman-Kac formula. In so doing, I have left aside, among other things, his impact on our present understanding of quantum mechanics, the vast mathematical domain devoted to 'Dirac operators' [21], that may [22–24] but need not be restricted to a manifold, and his novel idea of a 'magnetic monopole', which is closely related to the Chern class c_1 [25]. It is fair to expect that Dirac's collected works (1924–1948) will remain a source of inspiration for both mathematicians and physicists, be they active researchers or historians of science.

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