Problemen

| Problem Section

Problem 27 (L. Bleijenga)
Let L be a Latin square of order n. Show that any matrix $A \subset L$ of order $a \times b$ with $a + b = n + 1$ contains all elements $1, 2, \dots, n$.

Problem 28 (H. van den Berg)
For integers k, m, n and a prime number $p \geq 5$ show that if $(k^2 - mn)^p + (m^2 - kn)^p + (n^2 - km)^p = 0$, then p divides all three numbers $k^2 - mn, m^2 - kn, n^2 - km$.

Problem 29 (Lute Kamstra, open problem)
Let $n \in \mathbf{N}$, $h \in \mathbf{N}_0$ and let A be a subset of $\{1, 2, \dots, n + h\}$ of size n. Count the number of bijective maps $\pi : \{1, 2, \dots, n\} \to A$ such that $k \leq \pi(k) \leq k + h$ for all $1 \leq k \leq n$.

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Problem 21
Suppose that E^n is a finite-dimensional (real) vector space of dimension > 2 and that f, g are quadratic forms on E^n such that $f(x) = g(x) = 0$ implies that $x = 0$. Show that there are real numbers a, b such that $af + bg$ is positive definite.

Solution The solution is taken from a paper of E. Calabi. Consider the map $E^n \to \mathbf{R}^2$ defined by $x \to (f(x), g(x))$, which maps lines onto lines. Hence, this induces a map $F \colon P^{n-1} \to P^1$ between (real) projective spaces. The preimage of a point $(a, b) \in P^1$ is a quadric $(af + bg)(x) = 0$, which is a closed and connected subset of P^{n-1}. Since P^1 has fundamental group $\mathbf{Z}$ and P^{n-1} has fundamental group $\mathbf{Z}_2$, the map F can be lifted to $\tilde{F} \colon P^{n-1} \to \mathbf{R}$. If F were surjective, then there would be a point $(a, b) \in P^1$ such that $\tilde{F}$ maps onto two or more preimages of (a, b), contradicting that $(af + bg)(x) = 0$ is connected. So there exists an $(a, b) \in P^1$ which is not in the image of F. Then either $(af + bg)(x) < 0$ or $(af + bg)(x) > 0$ for all nonzero $x \in E^n$. Replacing (a, b) by $(-a, -b)$, if necessary, this gives a positive definite quadratic form $af + bg$.

The number 111001100000110101 is a square in base 5. In the following problems an n-binary number stands for a number that, written in base n, consists of digits 0 and 1 only, ending with a 1.

Problem 22
Prove that there are infinitely many 4-binary squares and 3-binary cubes with more than N digits equal to 1, for any natural number N.

Solution The following solutions were given by Aad Thoen. Observe that $a = 4^{2k+1} + 4^{k+1} + 1$ is a square for any natural number k. Now if x is a 4-binary square then so is ax, for sufficiently large k. The solution for 3-binary cubes is very neat. Consider the 3-binary numbers

$$x = \sum_{i=0}^{2n} 3^i \text{ and } y = \sum_{i=1}^{n} 3^{2i}.$$

Then $x = \frac{1}{2}(3^{2n+1} - 1)$ and $y = \frac{1}{8}(3^{2n+2} - 3^2)$. One verifies that $x^3 = 1 + (3^{4n+1} + 1)y$, which is a 3-binary number with $2n + 1$ digits equal to 1.

Problem 23 (Open problem)
Are there 3-binary squares with more than N digits equal to 1, for any natural number N?

Solution This problem remains open.

Solutions to the problems in this section can be sent to the editor — preferably by e-mail. The most elegant solutions will be published in a later issue. Readers are invited to submit general mathematical problems. Unless the problem is still open, a valid solution should be included.

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