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Algebraization of topology

It is aphoristically said that mathematics consists mainly of three basic disciplines: algebra, analysis and geometry, and the rest is their applications. However, in today's mathematics, the interplay of these disciplines is so intertwined and they are blended into one another to such an extent that it has become almost impossible to draw a line of demarcation between them. Each intrudes very often on territory of the others to give rise to new disciplines and in turn gets greatly stimulated in its own growth. Applications of topology (which is essentially analysis) to algebra and geometry have changed their entire fabric beyond recognition.

In this article, we intend to demonstrate how problems which are basically topological in nature can be extended, raised to new heights and even finally solved by employing algebraic methods, a process which we call 'algebraization of topology'. Although examples are available in abundance throughout mathematics, see [14], we have chosen four for discussion taking into account their importance, elegance and appeal to a general reader. They are:

1. Automatic Continuity
2. Stone-Weierstrass Theorems
3. The Closed Ideal Problem
4. Artin's Theory of Braids

In our exposition, we shall stress on laying bare the essential connections between these (without proof), particularly the role played by algebraic methods. We shall also take the reader to some of the frontiers of the literature on problems connected with these examples, except in case of example 4, the algebraic ramifications of which can be found in [4], [5], [12] and [28]. We include it here because of its fascination as one of the most simple classical examples of algebraization of topology.

Preliminaries

We explain some of the terminology that we shall use in the sequel. Our starting point is the concept of a vector space which is the 'absolute zero' for functional analysts. A vector space X is called a normed linear space if with each vector x in it there is associated a real number $||x||$, called the norm of x, such that $||x|| \geq 0$ and $||x|| = 0$ if and only if $x = 0$, $||\alpha x|| = |\alpha|\,||x||$ for each scalar α and $||x+y|| \leq ||x|| + ||y||$ for all x, y in X. Banach defined the concept of a normed linear space for the first time in his doctoral thesis in 1922 and thus laid the foundation of functional analysis. In fact, every normed linear space X is a metric space with the metric defined by $\mathrm{d}(x,y) = ||x-y||$, and hence a topological space. Thus what Banach did more significantly, by defining a normed linear space, was that he imposed a topological structure on a vector space, which is purely an algebraic structure. This is what we could term as the 'topologization of algebra'. A Banach space is a normed linear space which is complete (as a metric space). In fact, Banach himself calls complete normed linear spaces 'espaces du type $\mathcal{B}$' in a foot-note in his famous book [8], the very first monograph on functional analysis.

A Banach space X endowed with the additional structure of an inner product $\langle x,y\rangle$ such that the norm is related with the inner product by the equality $\langle x,x\rangle = ||x||^2$ is called a Hilbert space. A Banach algebra is a Banach space $\mathcal{A}$ which is also an algebra with identity 1 such that $||xy|| \leq ||x||\,||y||$ and $||1|| = 1$. A Banach algebra $\mathcal{A}$ is commutative if $xy = yx$ for all x, y in A. If $\mathcal{B}$ is a subalgebra of $\mathcal{A}$, then the set of all elements of $\mathcal{A}$ which commute with each element of $\mathcal{B}$ is also an algebra, called the commutant of $\mathcal{B}$ and denoted by $\mathcal{B}'$. The double commutant $\mathcal{B}''$ of $\mathcal{B}$ is defined by $\mathcal{B}'' = (\mathcal{B}')'$.

To discuss examples 1 and 2, we shall need some results from the theory of C^*-algebras, while for discussing example 3, we shall concentrate on by far the most important C^*-algebras $B(H)$ of all bounded linear operators on a Hilbert space H. A mapping $x \to x^*$ defined on a Banach algebra $\mathcal{A}$ is called an involution on $\mathcal{A}$ if:

i. $(ax+by)^* = \bar{a}x^* + \bar{b}y^*$
ii. $(xy)^* = y^*x^*$
iii. $(x^*)^* = x$ for all $x, y \in A$ and $a, b \in \mathbf{C}$.

A C^*-algebra is a Banach algebra $\mathcal{A}$ with an involution $x \to x^*$ such that

$$||x^*x|| = ||x||^2 \quad \text{(the $*$-quadratic norm equality).}$$

The concept of a C^*-algebra provides an example of a perfect diffusion of algebra, analysis and geometry. It borrows the $*$-algebra structure from algebra, completeness from analysis and the $*$-quadratic norm from geometry. In fact, the three aspects are so tightly fitted in the $*$-quadratic norm equality that any change in one aspect has automatic reflections on the other two. A C^*-subalgebra of the C^*-algebra $B(H)$ is called a von Neumann algebra if it is closed in the weak-operator topology. We have Dixmier [16] and Zhu [45] as good references on the subject.

Automatic continuity

If T is a mapping (transformation) of a metric space X into a metric space Y, then T may be continuous at certain points of X without being continuous on the whole of X (unless X consists of a single point). However, if X and Y are normed linear spaces, we have the following result on automatic continuity in elementary functional analysis:

Theorem. *Let X and Y be normed linear spaces and $T : X \to Y$ a linear transformation. If T is continuous at a point, then T is automatically continuous, that is, it is continuous at every point of X.*

It is easy to see that it is the linearity of T, purely an algebraic property, which determines the continuity of T once it is continuous at a single point.

We shall, however, concentrate on automatic continuity in Banach algebras. As a very special case, we consider C^*-algebras $\mathcal{A}$ and $\mathcal{B}$. A homomorphism $\psi : \mathcal{A} \to \mathcal{B}$ is called a C^*-homomorphism if $\psi(x^*) = \psi(x)^*$ for all $x \in \mathcal{A}$ and $\psi(1) = 1$. We have the following well known result; see [16] or [45].

Theorem. *Every C^*-homomorphism $\psi : \mathcal{A} \to \mathcal{B}$ is automatically continuous. In fact $||\psi(x)|| \leq ||x||$ for all $x \in \mathcal{A}$ and thus ψ is actually a contraction.*

For general Banach algebras nothing like this last theorem holds. But the commutativity of Banach algebras $\mathcal{A}$ and $\mathcal{B}$ and the semi-simplicity of $\mathcal{B}$ (both algebraic properties) make all the difference. A Banach algebra is called semi-simple if its radical (the intersection of all its maximal ideals) is $\{0\}$. We have the following theorem.

Theorem. *If ψ is a homomorphism of a commutative Banach algebra $\mathcal{A}$ into a semi-simple commutative Banach algebra $\mathcal{B}$, then ψ is automatically continuous.*

Kaplansky conjectured in 1950 that this theorem holds even if $\mathcal{A}$ and $\mathcal{B}$ are non-commutative. This remained an open question till 1967 when B.E. Johnson [23] proved a slightly weaker result:

Theorem (Johnson). *If $\mathcal{A}$ and $\mathcal{B}$ are Banach algebras, $\mathcal{B}$ semi-simple, then any onto homomorphism $\psi : \mathcal{A} \to \mathcal{B}$ is automatically continuous.*

As a particular case of this theorem, one can obtain the following conclusion which has many important applications [6].

Theorem. *Every involution on a semi-simple Banach algebra is continuous.*

B. Aupetit [6] has recently given an interestingly elegant short proof of an extension of Johnson's theorem via subharmonic functions. Can we replace an 'onto' homomorphism by an 'into' homomorphism in Johnson's theorem? No one knows!

Kaplansky's Conjecture. *If ψ is a homomorphism of a Banach algebra $\mathcal{A}$ into a semi-simple Banach algebra $\mathcal{B}$, is ψ continuous?*

Even the following problem continues to be unsolved.

Problem. *Let ψ be a homomorphism of a Banach algebra $\mathcal{A}$ into a semi-simple Banach algebra $\mathcal{B}$ with its range dense in $\mathcal{B}$. Is ψ continuous?*

There are partial solutions known in which the denseness of the range of ψ with some additional conditions guarantee the continuity of ψ, but in most of these cases ψ turns out to be onto, and hence continuous by Johnson's theorem. A typical example is the following result (see [6]).

Theorem. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras, $\mathcal{B}$ semi-simple. If $\psi : \mathcal{A} \to \mathcal{B}$ is a homomorphism such that its range is dense and has at most countable codimension in $\mathcal{B}$, then ψ is continuous.*

For further references on automatic continuity, see Sinclair [36] and Dales [15]; see also [7].

Lastly, we would like to mention yet another important result for von Neumann algebras, the so called double commutant theorem, which provides a very neat and clean yet quite deep example of algebraization of topology [45].

Theorem (The Double Commutant Theorem). *A C^*-subalgebra $\mathcal{A}$ of $B(H)$ is a von Neumann algebra if and only if $\mathcal{A} = \mathcal{A}''$.*

Stone-Weierstrass theorems

Every real polynomial is a continuous function but the converse is not true. The Weierstrass approximation theorem says, nevertheless, that continuous functions are not essentially far away from polynomials. Precisely speaking, for a closed interval $[a, b]$, the set $\mathcal{P}[a, b]$ of all polynomials is dense in the set $C[a, b]$ of all continuous functions defined on $[a, b]$. The crux of the generalization of this result by Stone is to replace the closed interval $[a, b]$ by a compact Hausdorff space X. But the problem is that to talk of a polynomial over X does not make sense. What Stone actually did was first to visualize and paraphrase Weierstrass theorem in terms of algebras. The process gives one of the most striking examples of algebraization of topology.

Firstly we observe that a polynomial is a finite linear combination of functions $1, x, x^2, x^3, \ldots$ In other words, the two functions 1 and x generate the algebra $\mathcal{P}[a, b]$, that can be regarded as a subalgebra of the algebra $C[a, b]$, which is actually a Banach algebra with sup norm. Since $\overline{\mathcal{P}[a, b]}$, the closure of $\mathcal{P}[a, b]$ in $C[a, b]$, is also an algebra, we see that $\overline{\mathcal{P}[a, b]}$ is the closed subalgebra of $C[a, b]$ generated by the functions 1 and x. The Weierstrass theorem states precisely that $\overline{\mathcal{P}[a, b]} = C[a, b]$. This algebraic version of Weierstrass theorem gives meaning to the question: "Can the closed interval $[a, b]$ in Weierstrass theorem be replaced by a suitable topological space X?" By closely examining the characteristic properties of functions 1 and x (the first is a constant function and the second separates points in $[a, b]$) and that $[a, b]$ is a compact Hausdorff space, Stone [37] proved the following generalization of Weierstrass theorem.

The Stone-Weierstrass Theorem. *Let X be a compact Hausdorff space and $C(X, \mathbf{C})$ the algebra of all complex continuous functions defined on X. If $\mathcal{A}$ is a closed subalgebra of $C(X, \mathbf{C})$ such that:*

i. it contains a non-zero constant function,
ii. separates points in X, and
iii. contains the conjugate $\overline{f}$ of each its function f,

then $\mathcal{A} = C(X, \mathbf{C})$.

Several remarks are in order. Firstly, we refer to Rudin [34] and Aupetit [6] for certain generalizations of the theorem which we shall not discuss. Secondly, we refer to Kaplansky [26] for an alternative proof of the theorem which is based on partitions rather than lattice-theoretic methods as given by Stone [37]. The two proofs are more or less of the same length, but while one could work with partitions in case of non-commutative algebras, with lattices this is not possible. Thirdly, as Weierstrass theorem occupies a salutary position in classical analysis, the importance of the Stone-Weierstrass theorem in modern analysis cannot be overemphasized. It is not surprising that its generalizations, extensions, unsolved problems associated with it and their partial solutions and variants are still attracting the attention of analysts. Fourthly, its generalization to locally compact Hausdorff spaces is rather routine as we shall see later, a proof which is still not so popular in the literature. We discuss here in brief its extensions to C^*-algebras.

As $C(X, \mathbf{C})$ is a commutative C^*-algebra, a natural question that arises is the following: 'If $\mathcal{A}$ is a C^*-algebra, not necessarily commutative, and $\mathcal{B}$ is a C^*-subalgebra of $\mathcal{A}$, under what conditions on $\mathcal{B}$ does $\mathcal{A}$ equal $\mathcal{B}$?' In order to examine certain answers to this question, we need to introduce some terminology.

An element x in a C^*-algebra $\mathcal{A}$ is positive ($x \geq 0$) if there exists an element $y \in \mathcal{A}$ such that $x = yy^*$. A functional ψ on $\mathcal{A}$ is said to be 'positive' if $\psi(x) \geq 0$ for all $x \geq 0$ in $\mathcal{A}$ and a 'state' on $\mathcal{A}$ if ψ is positive and $\psi(1) = 1$. The set $S(\mathcal{A})$ of all states on $\mathcal{A}$ is a subset of the closed unit ball in $\mathcal{A}^*$, the dual space of $\mathcal{A}$. $S(\mathcal{A})$ with the W^*-topology induced by the W^*-topology of $\mathcal{A}^*$ is called the 'state space' of $\mathcal{A}$. The state space $S(A)$ is a convex compact Hausdorff space, and hence has a non-empty set of extreme points. As a matter of fact, by the Krein-Milmann theorem $S(A)$ is the W^*-closed convex hull of the set of its extreme points. An extreme point of the state space $S(A)$ is called a 'pure state' of $\mathcal{A}$. We shall denote by $P(\mathcal{A})$ the W^*-closure of the set of extreme points of $\mathcal{A}$ together with $\{0\}$ and call it the 'pure state space' of $\mathcal{A}$.

Now if $\mathcal{A}$ is commutative, then $\mathcal{A}$ is isomorphic to the algebra of W^*-continuous functions on the set of pure states on $\mathcal{A}$. In fact, if $x \in \mathcal{A}$, then x corresponds to the function $\psi \to \psi(x)$, where ψ is a pure state on $\mathcal{A}$. As the set of pure states on $\mathcal{A}$ is a compact Hausdorff space, it follows by Stone-Weierstrass theorem that if $\mathcal{B}$ is a C^*-subalgebra of $\mathcal{A}$ which separates points in the set of pure states on $\mathcal{A}$, then $\mathcal{B} = \mathcal{A}$. Whether this is true even when $\mathcal{A}$ is noncommutative is an open question known as 'the Stone-Weierstrass problem'.

The Stone-Weierstrass Problem. *If $\mathcal{A}$ is a C^*-algebra, not necessarily commutative, and $\mathcal{B}$ is a C^*-subalgebra of $\mathcal{A}$ which separates the pure states of $\mathcal{A}$, is $\mathcal{B} = \mathcal{A}$?*

An affirmative answer to this question will provide a noncommutative extension of the Stone-Weierstrass theorem. In 1960 Glimm [21] obtained a partial solution of this problem by showing that it will suffice if $\mathcal{B}$ separates $\mathcal{P}(\mathcal{A})$, the pure state space of $\mathcal{A}$.

Theorem (Glimm). *If $\mathcal{A}$ is a C^*-algebra, not necessarily commutative, and $\mathcal{B}$ is a C^*-subalgebra of $\mathcal{A}$ which separates the pure state space of $\mathcal{A}$, then $\mathcal{B} = \mathcal{A}$.*

Before we close this section and suggest literature for further reading on the subject for which there is no scope to discuss in this article, we make a few observations. In fact, if $\mathcal{A}$ is commutative, then the set of pure states of $\mathcal{A}$ is W^*-closed in $\mathcal{A}^*$. Therefore, in the non-commutative case Glimm assumed a more stringent condition as suggested by Kadison that $\mathcal{B}$ separates not only the set of pure states of $\mathcal{A}$ but the pure state space $P(\mathcal{A})$ of $\mathcal{A}$ and proved the non-commutative C^*-algebra extension of the Stone-Weierstrass theorem by using some unpublished work of Kadison; see Dixmier [16].

The Stone-Weierstrass problem can be stated in another equivalent form. Kadison [25] proved that if $\mathcal{A}$ is a C^*-algebra, then there is a one-one correspondence between the pure states of $\mathcal{A}$ and the set $\mathcal{M}$ of maximal modular left ideals of $\mathcal{A}$. We say that a C^*-subalgebra $\mathcal{B}$ of $\mathcal{A}$ separates $\mathcal{M}$ if for $I, J \in \mathcal{M}$, we have $I \neq J \Leftrightarrow I \cap \mathcal{B} \neq J \cap \mathcal{B}$. Since if $\mathcal{A}$ is commutative and $\mathcal{B}$ separates $\mathcal{M}$, then $\mathcal{B} = \mathcal{A}$, the Stone-Weierstrass problem takes the following form.

The Stone-Weierstrass Problem, alternative form. *If $\mathcal{A}$ is a C^*-algebra, not necessarily commutative, and $\mathcal{B}$ is a C^*-subalgebra of $\mathcal{A}$ which separates $\mathcal{M}$, is $\mathcal{B} = \mathcal{A}$?*

Kaplansky [26] studied so-called CCR algebras. A C^*-algebra $\mathcal{A}$ is called a CCR algebra if every irreducible representation of $\mathcal{A}$ consists of completely continuous operators. He solved the Stone-Weierstrass problem in the affirmative in case $\mathcal{A}$ is a CCR algebra. In fact, his argument can he extended to achieve the same result in case $\mathcal{A}$ is a GCR algebra (a C^*-algebra with a composition series $\{I_\alpha\}$ of ideals such that $I_{\alpha+1}/I_\alpha$ is CCR). Although Glimm's proof is of genuine ingenuity, its bulk is devoted to use the ideas developed in [26]. Lastly we observe that the Stone-Weierstrass theorem has an easy extension to the algebra $C_0(X)$ of complex-valued continuous functions on a locally compact Hausdorff space X vanishing at infinity. In fact, each maximal modular ideal of $C_0(X)$ is hinged at some point $x \in X$, i.e. consists of those functions in $C_0(X)$ which vanish at x. Now it suffices to observe that a C^*-subalgebra $\mathcal{B}$ of $C_0(X)$ separates points in X if and only if it separates its maximal ideals and consequently the Stone-Weierstrass theorem holds for $C_0(X)$.

We suggest further reading on the subject. Dixmier [16] provides a good exposition of the work of Kaplansky [26] and Glimm [21] with simplification of their arguments. Akemann [1] was first to obtain some partial solutions of the general Stone-Weierstrass problem. In [2], Akemann and Anderson have given a survey of the various approaches attempted by a number of authors up to 1982 to solve the problem in the light of the general framework developed by Akemann [1]. Some results on factorial Stone-Weierstrass problem for separable C^*-algebras were obtained by Anderson and Bunce [3]. Earlier in 1970 Sakai [35] had solved the problem in case $\mathcal{A}$ is separable and $\mathcal{B}$ is nuclear. Longo [27] and Popa [31] have independently obtained a solution of the factorial Stone-Weierstrass problem for separable C^*-algebras. For recent complements to Stone-Weierstrass theorem, see Brown [13].

The closed ideal problem

Here we discuss a problem in operator theory which remained open for more than four decades (1940-1984) and in the solution of which algebraic methods have played a great role, as we shall show. Let T be an operator in $B(H)$, where H is an infinite-dimensional separable complex Hilbert space. A closed subspace M of H is called invariant under T if $T(M) \subset M$ — $\{0\}$ and H are trivially invariant under T. M is called non-trivial if $M \neq \{0\}$ and $M \neq H$. Does every operator T have a non-trivial invariant subspace? This is the so-called 'invariant subspace problem', which is still open. No one knows as to who raised the problem for the first time. Perhaps it arose naturally amongst mathematicians after the unpublished discovery by von Neumann that every non-zero compact operator has a non-trivial invariant subspace. The problem has been solved for Banach spaces; see Read [32], [33], Enflo [19] and Beauzamy [9], [10].

Consider a sequence $w = \{w_n\}_{n=0}^{\infty}$ of positive real numbers such that $\sup\{w_{n+1}/w_n\} < \infty$. Define $l^1(w)$ as the space of all complex sequences $x = \{x_0, x_1, \ldots\}$ with

$$||x|| = \sum_{n=0}^{\infty} |x_n| w_n < \infty.$$

Then $l^1(w)$ is a Banach space isometrically isomorphic to l^1. The (forward) shift operator T on $l^1(w)$ is defined by

$$Tx = \{0, x_0, x_1, \ldots\}.$$

Since $\{w_{n+1}/w_n\}$ is a bounded sequence, T is a bounded operator on $l^1(w)$. The invariant subspaces of T will simply be called the invariant subspaces of $l^1(w)$. The subspaces $M_k(w)$ defined by

$$M_k(w) = \{x \in l^1(w) : x_n = 0, n < k\}, \quad 0 \leq k < \infty,$$

are obviously invariant subspaces of $l^1(w)$. These are called the standard invariant subspaces of $l^1(w)$. Whatever is said so far about $l^1(w)$ holds true about the Banach spaces $l^p(w)$, $1 \leq p < \infty$, defined accordingly. Characterization of standard invariant subspaces of $l^p(w)$ has been a fascinating subject for many mathematicians; see for example references [18], [29], [42]–[44].

If the weight sequence w_n satisfies also an additional condition:

$$w_{m+n+1} \leq C w_m w_n \quad \text{for all } m, n$$

and a constant $C > 0$, then $l^1(w)$ is a Banach algebra and the invariant subspaces of $l^1(w)$ are actually its closed ideals. The Banach spaces $l^p(w)$, $p > 1$, become Banach algebras under more stringent conditions on the weight sequence $\{w_n\}$. See [22]. In either case, the ideals $M_k(w)$ are called the standard closed ideals of these algebras. The algebras $l^1(w)$ are the special cases of more general weighted Banach algebras first studied independently by Beurling [11] and Gelfand [20], while the algebras $l^p(w)$, $p > 1$, are closely related to the work of Wermer [40].

A Banach algebra $l^p(w)$ is called radical if the weight sequence $\{w_n\}$ satisfies the condition $\lim_{n\to\infty} w_n^{1/n} = 0$.

Problem. *Is every closed ideal in a radical Banach algebra $l^p(w)$ an $M_k(w)$? Or, equivalently, does there exist a radical Banach algebra $l^p(w)$ with a non-standard closed ideal?*

This is called the 'closed ideal problem'. The problem for radical $l^p(w)$ algebras was first raised by Shilov around 1940 and remained unanswered till 1984 when Thomas [38] constructed an example of an algebra which has a non-standard closed ideal. He further showed in 1985 that his construction can be extended to $l^p(w)$-algebras for $p > 1$ [39]. Nikolski [30] had earlier claimed to have constructed an example of an $l^1(w)$-algebra having a non-standard closed ideal, but his construction was discovered to be erroneous. In fact, the weight sequence w that he constructed did give an example of a non-standard invariant subspace in the Banach space $l^1(w)$, but it failed to be an algebra weight sequence.

The crux of Thomas' construction can be explained as follows: let us denote by $\overline{(\mathcal{A}x)}$ the closed ideal generated by an element x in $\mathcal{A} = l^1(w)$. Thomas actually constructed for the first time a Banach algebra $\mathcal{A}$ such that:

i. $\overline{(\mathcal{A}x)} = \mathcal{A}$, that is, x is a generator of $\mathcal{A}$, and
ii. $\mathcal{A}$ has just one maximal ideal containing properly a closed ideal which is not contained in $\overline{(\mathcal{A}x^2)}$.

On the other hand, it is interesting to ask for conditions on the weight sequence w which determine that all the closed ideals of $l^1(w)$ are standard ones only. There is good literature on the subject and we refer to [41] as a source of information.

Artin's theory of braids

Here is an example of algebrization of topology in which continuous deformations of a system are determined by purely algebraic methods involving elementary group theory. We refer to Birman [12] and Moran [28] for a good account of the subject. See also Kac and Ulam [24].

We consider two parallel straight lines A and B in space which are identically orientated and choose an equal number of points on A and B, say a_1, a_2, a_3, a_4 on A and b_1, b_2, b_3, b_4 on B, as shown in figure 1. Each a_i is connected to a unique b_j through a curve c_i. The curves c_i may twist and wind in space subject to the condition that the projection of each c_i in the (A, B)-plane remains monotone, a property that guarantees that the distance of a bug from the line A as it moves from the point a_i to the point b_i along the projection of c_i is monotonically increasing. The system consisting of the two lines A and B along with the curves c_i is called a 'weaving pattern'.

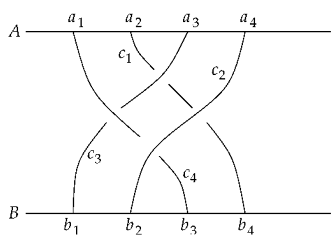

Figure 1

We make the following postulates on deformations of a weaving pattern:

i. Although the distance between the lines A and B may vary, they remain parallel with the same orientation.

ii. The points a_i and b_i can move arbitrarily along their respective lines irrespective of the distance between two consecutive points but their order remains unchanged.
iii. The curves may be contracted or stretched but the projection of each c_i in the (A, B)-plane under any such deformation continues to be monotone. (This guarantees that the multiplication of two weaving patterns as introduced below is well-defined.)
iv. No two c_i's intersect during any deformation.

We call two weaving patterns 'equivalent' if each can be obtained from the other by a deformation having properties (i)–(iv). It is easy to see that this equivalence between two weaving patterns is actually an equivalence relation. We define a *braid* as an equivalence class of weaving patterns. Now the question is how to identify two braids.

Fundamental problem. *Which two braids are identical?*

Equivalently, we may ask the question: 'Which two weaving patterns are equivalent?' In order to solve the problem, we first define a binary operation on the set of weaving patterns. Given two weaving patterns W_1 and W_2, we define $W_1 \circ W_2$ as follows: put W_2 below W_1 so that the line A of W_2 lies on the line B of W_1 and the points a'_1, a'_2, a'_3, a'_4 coincide respectively with b_1, b_2, b_3, b_4 as shown in figure 2.

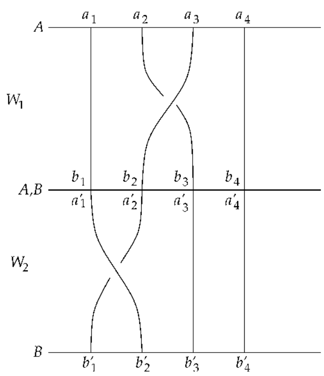

Figure 2

Now defuse W_1 and W_2 into $W_1 \circ W_2$ by removing the line in the middle. The result is figure 3.

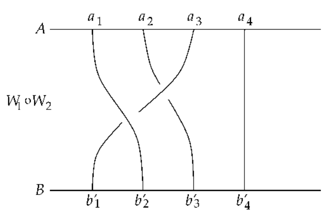

Figure 3

It is obvious that this binary composition is associative. Also, the trivial weaving pattern I serves as the identity element for the binary operation. The inverse of a weaving pattern, say of W_1, is obtained by interchanging the positions of lines A and B (see figure 4). We see that $W_1 \circ W_1^{-1} = W_1^{-1} \circ W_1 = I$ and have

Theorem. *The set G of all weaving patterns is a group with respect to the binary operation defined above.*

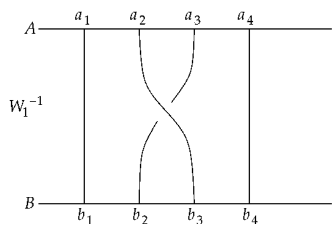

Figure 4

Now consider the three weaving patterns W_1, W_2 and W_3 with their inverses. They are pictured in figure 5.

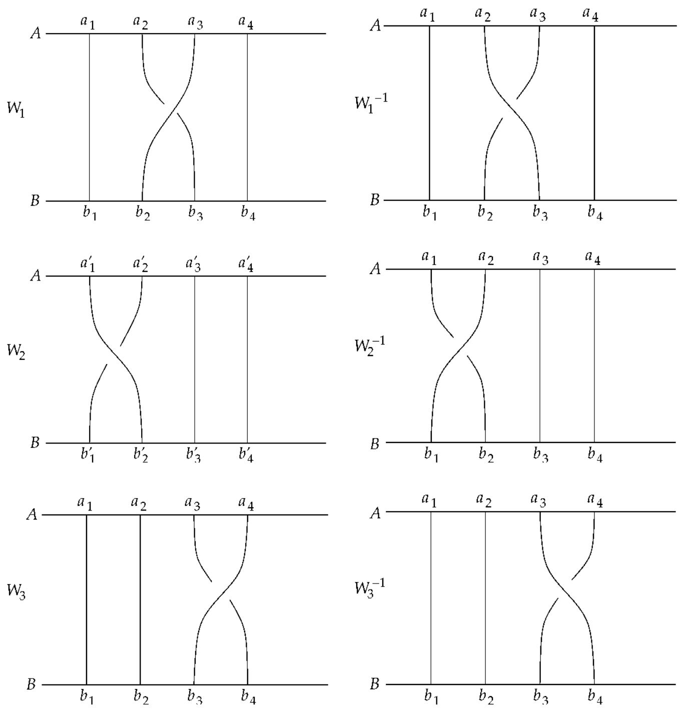

Figure 5

It's not hard to see that each weaving pattern in the group G can be represented as a product of these six elements. For example, if W is as in figure 6, then we see that $W = W_2 W_3^{-1} W_1$.

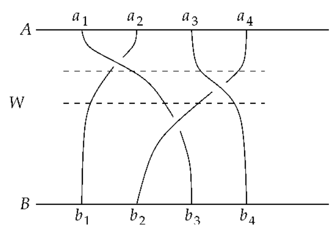

Figure 6

This shows that for each W in G, we have $W = D_1 D_2 \cdots D_n$ where each D_i, $i = 1, 2, \ldots, n$, is one of these six elements. We call the representation $D_1 D_2 \cdots D_n$ a word and observe that each weaving pattern W in G can be represented as a word. However, we see that this representation is not unique as different words may represent the same element of G. For example, we have $W_1 W_1^{-1} = W_2 W_2^{-1} = W_3 W_3^{-1} = I$.

Thus we finally conclude:

Theorem. *Any two weaving patterns are equivalent if and only if the corresponding words represent the same element of G.*

This work of E. Artin [4], [5] amply illustrates how a problem essentially topological in nature can be solved by employing methods which are purely algebraic.

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